DECISION TREES WITH INDEPENDENT STOCHASTIC ACTIVITY DURATIONS

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DECISION TREES WITH INDEPENDENT

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DEDICATION

I sincerely dedicate this thesis to the most important woman in my life, Linda.

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SUMMARY

This thesis provides an implemented method of solving the generalized stochastic decision tree problem, where the generalization consists of allowing the durations of activities to be independent random variables. Such problems arise naturally, for example, in deciding how much overcapacity to provide in a facility designed for uncertain growth in demand.

A FORTRAN-IV computer program for generalized stochastic decision trees is provided and documented. Compared with existing decision tree algorithms, its extensions are (1) the durations of activities may be random variables having any of several probability distributions, (2) rewards or costs may be incurred uniformly throughout a variable-duration activity, (3) variances of the present worth along solution paths are reported, taking into account the joint (independent) variabilities of timing and amount, and (4) mean/variance approximation of variable activity durations is allowed. Extensions 1 and 2 allow consideration of such things as extended construction costs and delays in operating profits until completion of a project; extension 3 allows joint consideration of profitability and risk; extension 4 simplifies data gathering and modeling.

Proofs of extensions 1, 2, and 3 are presented. A series of experiments is presented to demonstrate the high accuracy of the mean/variance activity-duration approximation. It is shown that the expected cost of a wrong decision using the approximation is less than 0.009% for a wide variety of decision tree structures of practical interest where

interest rates are below 24% and the average duration of activities is below five years.

CHAPTER I

INTRODUCTION

1. Purpose of Research

The objective of the work reported in this thesis is to provide decision makers with a practical means for solving decision tree problems to maximize the expected present worth where the activities (arcs) are of uncertain duration.

It is assumed that the objective of the decision maker is to identify the optimal set of decisions giving paths having maximal expected present worth at an interest rate equal to the "minimum attractive rate of return." It is further assumed that the decision maker will want to calculate expected present worths for this set of optimal decisions and also for decision sets that are sub-optimal, and to calculate the variance of the present worth for each decision set.

To meet these objectives, conventional decision tree methods [31, 32,50,21] require cash flow amounts to be independent random variables with known means and variances, and activity durations to be fixed.

The work reported in this thesis provides extensions to conventional decision tree methods. Specifically, we relax the requirement that activity durations be fixed, requiring instead that activity durations be independent of each other and of the independent cash flow amounts. With known probability distributions of activity durations, exact solutions are provided. With known means and variances of activity durations where the distribution is unknown, approximate solutions are provided. It is shown that the errors associated with using the approximation for reasonable decision problems are negligibly small.

2. Background

This section is intended to provide the reader with a definition of terms and fundamental relationships used elsewhere in this thesis. The section is divided into two parts. The first part defines present worth in terms of both discrete and continuous times. A definition of the minimum attractive rate of return is also provided. In the second part a decision tree is defined and is related to a probability network. Both the linear programming and dynamic programming approach to solving decision trees is presented along with a definition of an optimal solution to a decision tree problem.

2.1 Present Worth Calculations with Variable Timing

Discounted cash flow methods, in which financial transactions occurring at different times are rendered commensurable by comparing them to lending/borrowing transactions at compound interest, are widely used. For example, Fremgen [14] found that 135 of 177 surveyed businesses use formal discounted cash flow methods to evaluate proposed capital investments.

Let worth have units of dollars, and let time have units of years. Let a concrete or abstract entity or state of affairs have a worth denoted by F at a discrete time denoted by n. Let there be a compound interest rate i having units of reciprocal years. The worth of the same entity or state of affairs at time zero (n units earlier than the time associated with F) is called its present worth, denoted P, and is defined by the fundamental equation of discounted cash flow:

$$P = F/(1+i)^n = F\beta^n = Fe^{-rn} \tag{1}$$

The single period discount factors $1/(1+i)$, β, and e^{-r}, will be used interchangeably. The quantity $r = \ln(1+i)$ is called the nominal continuous compound interest rate.

If we denote by F the worth of an entity or state of affairs at the discrete time n, we can define the present worth P of a set of these, each element of the set having a different time n, by sums of equations of the form of equation (1). Specifically, with $n \varepsilon \{0,1,2,\ldots,k\}$, we have

$$P = \sum_{n=0}^{k} F_n \cdot \beta^n \ . \tag{2}$$

Similarly, if we consider a time-ordered set of entities or states of affairs having worths F_η at times $t_\eta, t_{\eta+1} > t_\eta$, with epochs $\eta \varepsilon \{0,1,2,\ldots,k\}$, the present worth of the set is defined

$$P = \sum_{\eta=0}^{k} F_\eta \cdot \beta^{t_\eta} \ . \tag{3}$$

Equation (2) defines present worth for integer times, and equation (3) defines present worth for continuous times.

The motiviation for these definitions can be seen from their formal identity with the equations governing lending/borrowing transactions at compound interest. A loan of P dollars is exactly repaid at compound interest rate i by a series of repayments $\{F_n\}$ that satisfies equation (2). or by a series of repayments $\{F_\eta\}$ that satisfies equation (3).

If we assume there exists for a decision maker an interest rate i, called the "minimum attractive rate of return," at which funds can be freely lent or borrowed in the ordinary course of business, this assumption is tantamount to the assumption that the decision maker is indifferent to receiving a series of future payments versus the alternative of receiving at time zero the present worth of the series of future payments. The decision maker is said to consider P "equivalent" to $\{F_n\}$ or $\{F_\eta\}$.

Discounted cash flow methods in common use [50] follow directly from these assumptions. They proceed as follows. First, a set of mutually exclusive alternative actions is defined. For each action, all future consequences are classified exhaustively and mutually exclusively; typical examples of consequences include a required investment, a tax liability, control of an asset, entitlement to something of value, etc. If deterministic methods are in use, the next step is to assign a worth and a time to every consequence, and to combine (add together) coeval consequences so that there is one F_n for each time n or one F_η for each epoch η. Equation (1) or (2) is then solved to yield a P for each alternative action, and the action having the highest P is chosen.

If probabilistic methods are in use, the procedure is the same except that probability distributions and their parameters are assigned for worths and times, and the expected values of P are calculated by taking expectations or conditional expectations of equations (1) and (2), with the action having the highest $E\{P\}$ being chosen. Often [20,23,24,30,43] this maximization of the expected present worth is supplemented by examination of the variance or further moments, or characterization of the entire probability distribution of present worth; however, in the absence of accepted formal methods of using the further information, accepted practice is simply to report the supplemental information for intuitive use by the decision maker.

2.2 Decision Trees

A probability network is a graph whose nodes represent states and whose directed arcs represent possible transitions from state to state. A finite probability network can be completely specified by a list of its nodes, a list of its directed arcs, and a conditional probability for each possible transition. In a finite state space $S = \{\ldots,i,j,\ldots\}$ let there be N states or nodes. Let directed arcs or transitions (i,j) exist for some pairs of nodes $i \varepsilon S$ and $j \varepsilon S$, and let p_{ij} be the conditional probability of transition (i,j) given state i.

A probability decision network is a probability network that is incompletely specified by allowing some of the p_{ij} to be chosen to optimize some function. We confine attention here to maximizing the expected value of a sum of rewards r_{ij} each realized if the corresponding transition (i,j) occurs in a sample trajectory.

A decision tree is a probability decision network that is a tree; it contains one initial state or root node having no entering arc, a set T of terminating nodes having no leaving arcs, and a set S-T-s of ordinary nodes each having exactly one entering arc and two or more leaving arcs. For example, in Figure 1 node 0 is s, T contains nodes 3,4,5,6, and nodes 1 and 2 are ordinary. Since one arc enters each node except node s, its number of arcs is N-1.

In a sample trajectory of a decision tree the system starts in state s and transmits through ordinary states until it enters a terminating state, and a reward r_{ij} is collected for each transition (i,j) that occurs in the sample trajectory.

The decision problem in a decision tree is to choose all p_{ij} not fixed, constrained of course by $0 \le p_{ij} \le 1$ and $\sum_j p_{ij} = 1$, to maximize total expected reward. Non-terminating node i is a chance node if its p_{ij} are fixed, a choice node otherwise. In Figure 1 node 0 is a choice node and nodes 1 and 2 are chance nodes.

One algebraic representation of a decision tree is a linear program having a variable for each arc. Call this program the primal. Its dual linear program has a variable for each node; call this program the dual. Efficient solution procedures for decision trees manipulate the variables of the dual, but the primal is of theoretical interest in developing extensions to these methods.

In formulating the primal, the conditional probabilities p_{ij} are not convenient as primal decision variables. Let us define the corresponding unconditional probabilities x_{ij}, where x_{ij} represents the probability that a transition (i,j) occurs in a sample trajectory. Since r_{ij} is collected with probability x_{ij}, the total expected reward to be maximized is $\sum_J r_{ij} x_{ij}$, where J is the set of arcs (i,j) that exist in the tree. The probability that a non-terminating node j occurs in a sample trajectory is equal to x_{ij} and is also equal to $\sum_k x_{jk}$. For each fixed p_{jk}, the corresponding x_{jk} equals $p_{jk} x_{ij}$ by the definition of conditional probability. For each p_{jk} not fixed, the primal program's optimal solution chooses the corresponding x_{jk}. The primal program is:

$$\text{Maximize} \quad \sum_J r_{ij} x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_k x_{jk} - x_{ij} = 0 \quad , \; j \neq s, (j,k), (i,j) \varepsilon J \tag{2a}$$

$$\sum_k x_{sk} = 1 \tag{2b}$$

$$x_{jk} - p_{jk} x_{ij} = 0 \quad , \; p_{jk} \text{ fixed}, \; (j,k), (i,j) \varepsilon J \tag{2c}$$

$$x_{ij} \geq 0 \quad , \; (i,j) \varepsilon J \tag{2d}$$

A conventional decision tree is one in which for each node j, either the p_{jk} are given for all k (j a "chance node") or the p_{jk} are unspecified (j a "choice node"). If the unspecified p_{jk} are considered to be the decision variables, the optimal solution is known [37] to have $p_{jk} \varepsilon [0,1]$. It is customary to represent such problems graphically. For example, in the following solved decision tree, node 1 is a choice node and nodes 2 and 3 are chance nodes:

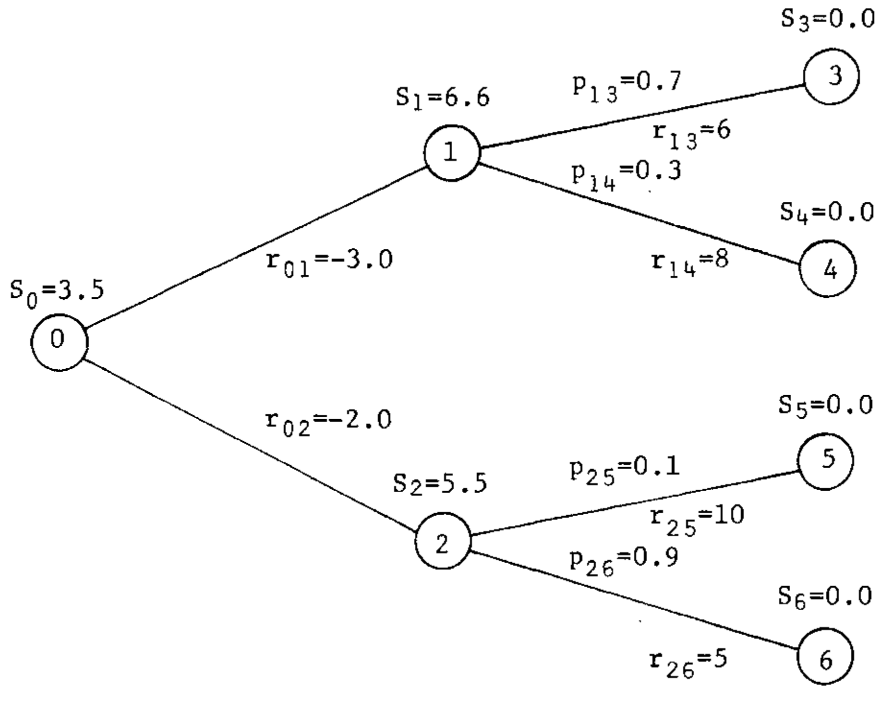


Figure 1. Representation of a Decision Tree

The well known dynamic programming or fold-back solution procedure [40] involves labeling each node with a "value" equal to the value of the dual variable of the linear programming formulation. Proceeding from right to left, the expression $S_i = \text{Max}\{p_{ij}(r_{ij}+S_j)\}$ is solved for each non-terminal node. The Max operator is suppressed for chance nodes, where p_{ij} are given. For choice nodes, p_{ij} are set equal to 1 for j that maximizes $(r_{ij} + S_j)$ and to 0 for other j.

This procedure is easily extended to include discounting. In the objective function, let r_{ij} be replaced by $\alpha_i r_{ij}$, where α_i is the present worth discount factor for node i; this factor discounts a cash flow occurring at the time of the event denoted by node i. Node zero is considered to occur at time zero. Each activity or arc (i,j) is considered to have a duration T_{ij}, so that the discount factor β for a specific node, say node k, is the product of the discount factors $\beta^{T_{oi}} \cdot \beta^{T_{ih}} \ldots \beta^{T_{jk}}$, where β is the discount factor defined in equation (1), section I.2.1, and the unique path to node k passes through arc (o,i),(i,h),...,(j,k). The expression for S_i then becomes

$$S_i = \text{Max}\{p_{ij}(r_{ij}+\beta^{T_{ij}} S_j)\} \tag{3}$$

and the solution proceeds as before. If arc (i,j) has a cash flow F at its start, a cash flow G at its end, and a uniform continuous cash flow $\bar{A}$ throughout its duration, then $r_{ij} = F + G\beta^{T_{ij}} + \bar{A}(1-\beta^{T_{ij}})/r$; that is, r_{ij} represents the present worth at the time of event i for all cash flows associated with activity (i,j).

This thesis concerns exact and approximate solutions for generalized decision trees which are defined such that the T_{ij} may be independent random variables, thus affecting present worths of costs, and in which costs of activities may depend on the T_{ij}.

3. Content Summary

In Chapter II, methods currently available for solving stochastic decision trees are discussed. These methods range from the classical methods of transforming the stochastic problem into a deterministic problem to the complex approach of convoluting Laplace Transforms of the functions of the random variables.

In Chapter III, a solution procedure is defined for solving stochastic decision trees with both termination cash flows and continuous uniform cash flows. This solution procedure provides for determination of the variance of the "optimal path" as well as the present worth. It is based on expressions for the expected value of the time dependent discount factor derived by Young and Contreras and on a variance relationship derived by Rosenthal.

In Chapter IV, an approximation derived by Young and Contreras is tested for accuracy when applied to both cash flow paths and the solution of decision trees. The expectation of a wrong decision is investigated along with the expected cost of selecting a sub-optimal decision set.

In Chapter V, a computer program to solve stochastic decision trees is described.

CHAPTER II

CURRENT METHODOLOGY

1. Methods for Solving Discounted Cash Flows with Random Time Intervals

A review of currently available methods for solving stochastic cash flow problems is presented in this chapter. Throughout this chapter, the random variable F_η is the magnitude of the cash flow associated with epoch η, and the random variable T_η is the time interval from epoch η to epoch $\eta+1$. An epoch is defined by the occurrence of a payment or the starting time of a uniform continuous series of payments.

1.1 Classical Methods

One popular method of preparing data for decision trees is to consider the most likely time estimate to be certain and solve the deterministic problem [31,32]. Under this approach, we replace $E\{\beta^T\}$ with β^M, where M is the mode of the time function. A variation is to set T equal to the mean or expected value of the time distribution, replacing $E\{\beta^T\}$ with β^μ, where $\mu = E\{T\}$. Two major objections may be raised with regard to the use of these methods; they are insensitive to the shape of the timing distribution for the occurrence of the cash flow, and the resultant answer does not yield any information as to the variablility of the result.

1.2 Beta Distribution Method

Another classical method proposes that the time periods under consideration be broken into small increments with probabilities of termination being associated with each sub-interval of the total activity time.

Under this approach, either a discrete distribution is used to characterize the timing of the cash flow or a continuous distribution is used where the probability of the flow occurring at the nth sub-interval is given by:

$$\int_{n-1}^{n} f(T)dT = F(t_n) - F(t_{n-1}) \tag{1}$$

where F(•) is the cummulative distribution function of the timing. This method requires that the form of the distribution be specified. To alleviate the need to specify the distribution type, Greer [15] proposed a method in 1970 based on the beta distribution.

Greer's method has the advantage that the distribution shape is flexible, conditional probabilities of occurrence are all monotone increasing, it has a discrete range, and will not take on negative values. For this method, the beta distribution is defined as follows:

$$f_T(t) = (\text{constant})(t-A)^c(B-t)^d \tag{2}$$

where A and B are the lower and upper bounds of the distribution, respectively, providing the beta distribution with a range between $f_T(A)$ and $f_T(B)$, and the constant is used to normalize the area under the curve. The random variable T is normalized such that $0 \le T' \le 1$ by setting

$$T' = (t-A)/(B-A) \tag{3}$$

and the mode, M, is normalized similarly for the same range by

$$M' = (M-A)/(B-A) \quad . \tag{4}$$

Once these values are established, c and d can be determined by equating the known values for the normalized mode, M', and the variance, V, to the appropriate parameter combinations for the beta distribution, namely:

$$M' = c/(c+d) \tag{5}$$

$$V = (c+1)(d+1)/[(c+d+2)^2(c+d+3)] \quad . \tag{6}$$

The assumption is made that the variance, V, is equal to c/36 based on the normalized range of T' with the assumption that the range covers six standard deviations. Solving the set of simultaneous equations (5) and (6) yields values for c and d. Due to the normalization of M' to values between 0 and 1, c is guaranteed to have at least one positive value. Multiple values signify alternate solutions. Each positive c yields a value for d. Finally the constant is computed in order to make the area under the distribution equal to unity. For non-integer c and d, this is not an easy process and is therefore circumvented by establishing an increment δ, based on the accuracy desired, to solve:

$$\int_{n}^{n+\delta} (\text{constant})(t-A)^c(B-t)^d \, dt, \tag{7}$$

$$n = A, A+\delta, \ldots, B-\delta, B$$

To eliminate analytical determination of the constant term, the integral expression is normalized over the range of possible values:

$$P\{n \le t \le n+\delta\} = \left[\int_{n}^{n+\delta} (t-A)^{c}(B-t)^{d}\,dt\right] \Big/ \left[\int_{A}^{B} (t-A)^{c}(B-t)^{d}\,dt\right] \quad (8)$$

$$n = A, A+\delta, \ldots, B-\delta, B$$

This provides a set of weights for discount factors to find the probability distribution of the discounted cash flow (present worth).

For determination of the integral expression (8), Greer developed a table relating discount factor values and areas under the curve of the beta distribution. This table enables the user to easily determine the probabilities associated with the $(B-A)/\delta$ intervals within sight accuracy.

The major disadvantage of Greer's method, as well as with any incremental probability method, is the large number of calculations necessary to evaluate each branch of a decision tree.

1.3 Monte Carlo Simulation

Perhaps one of the most popular methods for solving stochastic decision problems is through Monte Carlo Simulation. This technique (as applied to ordinary decision trees) is presented in [21] and also in [8] and [22]. Although these articles present the technique relative to deterministic event times, the application of the technique to stochastic event times is possible by considering time to be one of the random variables.

Once again, as with Greer's method, a major disadvantage is the large number of computations required to derive reasonably accurate solutions.

1.4 Transform Analysis Method

The transform analysis method, proposed by Perrakis and Henin [38] and Perrakis and Sahin [39], extends the work of F. Hillier [23] to producing a distribution of the net present value of a cash flow path where the intervals between successive cash inflows are independently distributed and independent of the magnitude of the cash inflows. The case of perfectly correlated cash inflows is also presented but will not be discussed here.

The Perrakis-Henin formulation is based on three assumptions and the fact that there exists a one-to-one correspondence between a function and its Laplace transform. The Laplace transform is defined by:

$$\int_0^\infty e^{-sx} f(x)\, dx \quad . \tag{9}$$

The three assumptions are:

1. All cash outflows occur at time zero. If it is desired to have a cash outflow occur at a time different from zero, the outflow must be discounted to its time zero value before beginning the analysis;
2. All cash inflows occur after time zero;
3. The number of cash receipts, k, is known, where k is an integer greater than or equal to one.

The formulation begins with the definition of net present worth:

$$P_n = \sum_{\eta=1}^{n} F_\eta \cdot \exp(-rt_\eta) \tag{10}$$

where F_η is the magnitude of the η-th cash flow and t_η is the time of occurrence of the η-th cash flow. The random variable T_η is defined as the difference between the time of occurrence of the η-th and $(\eta-1)$-th cash flow:

$$t_n = \sum_{\eta=1}^{n} T_\eta \tag{11}$$

This yields the following re-definition of the net present worth:

$$P_n = \sum_{\eta=1}^{n} F_\eta \cdot \exp(-r \cdot \sum_{m=1}^{\eta} T_m) \tag{12}$$

The time axis is then reversed in order to proceed from the last cash flow to the first cash flow. Under this arrangement:

$$P_n = \sum_{i=1}^{n} F_i \cdot \exp(-r \cdot \sum_{j=1}^{n} T_j) \tag{13}$$

S_i is defined as the magnitude of the process (or the net present worth after the i-th cash flow on the reverse time access). This yields the following set of recursive equations as we proceed through the cash flow path.

$$S_1 = F_1 \tag{14}$$

$$\vdots$$

$$S_{i+1} = S_i e^{-rT_i} + E_{i+1} \tag{15}$$

$$\vdots$$

$$S_k = S_k e^{-rT_k} \tag{16}$$

We define $B_{i+1}(F)$ as the cummulative distribution function for the magnitude of the (i+1)-th cash flow; $b_{i+1}(F)$ as the corresponding density function; and $g_i(T)$ as the density function of the time duration before the cash flow, or T_i. Given these definitions it is clear that:

$$B_{i+1}(F) \;=\; P\{S_{i+1} \le F\} \;=\; P\{S_i e^{-rT_i} \le F\} \tag{17}$$

$$= \; P\{S_i e^{-rT_i} \le F \mid T_i = \tau\}\; g_i(\tau) \tag{18}$$

summed over all time where τ is a particular time. This is an application of conditional probabilities where F_i and F_j are independent for all $i \neq j$, and T_i, T_j are independent for all $i \neq j$, and F_i, T_i are independent for all i. Hence:

$$P\{S_i e^{-rT_i} \le F\} \;=\; \int_0^{\infty} P\{S_i e^{-rT_i} \le F \mid T_i = \tau\}\; g_i(\tau)\, d\tau \tag{19}$$

Since $B_i(F) = P\{S_i \le F\}$ it follows that $B_i(Fe^{rT_i})$ is equal to $P\{S_i \le Fe^{rT_j}\}$. Hence:

$$P\{S_i \le Fe^{rT_i} \mid T_i = \tau\} \;=\; B_i(Fe^{r\tau}) \tag{20}$$

which is the cummulative distribution of T_i evaluated at τ. Since the future effective discounted cash value is proportional to the difference between the new and the old value, let $(F-\chi)$ represent the difference, summed over all possible previous values which are a function of χ, namely $B(\chi)$. This yields:

$$B_{i+1}(F) = \int_0^{\Phi}\left[\int_0^{\infty} B_i[(F-\chi)e^{r\tau}]g_i(\tau)\ d\tau\right]\ dB(\chi) \tag{21}$$

which when extended to all cash flows k yields:

$$P_k(F) = \int_0^{\infty} B_k(Fe^{r\tau})\ g_k(\tau)\ d\tau \tag{22}$$

The resulting set of recursive relationships for the net present worth in terms of the density functions associated with the magnitude of the cash flows and the times of occurrence of the cash flows as shown below:

$$b_1(F) = \int_F B_1(F)\, dF \tag{23}$$

$$b_{i+1}(F) = \int_0^{\Phi}\left[\int^{\infty} b_i[(F-\chi)e^{r\tau}]e^{r\tau}\ g_i(\tau)\ d\tau\right] b(\chi)\ d\chi \tag{24}$$

$$P_k = b_k = \int^{\infty} b_k(Fe^{r\tau})\ e^{r\tau}\ g_k(\tau)\, d\tau \tag{25}$$

This set of recursive relationships is then defined in terms of Laplace transforms in order to simplify the use of the superposition integral to convolute the many independent functions describing the arcs of the cash flow path. The superposition integral is defined by:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\ f_Y(z-x)\ dx \tag{26}$$

Signifying the Laplace transform with a bar placed over the function operator, we apply the following theorem of convolution:

$$\overline{(f*g)} = \overline{f} \cdot \overline{g} \quad , \tag{27}$$

i.e., the Laplace transform of the convolution of two functions is equal to the product of the Laplace transforms of the two functions. Application of this theorem and taking Laplace transforms of the functions in equations (23), (24) and (25) yields:

$$\overline{b_1}(F) \qquad = \text{direct Laplace transform of } b_1(F) \tag{28}$$

$$\overline{b_{i+1}}(F) \qquad = \overline{b_i}(s) \int_0^\infty \overline{b_i}(se^{-r\tau})\, g_i(\tau)\, d\tau \tag{29}$$

$$\overline{P_k} = \overline{b_k}(F) = \int_0^\infty \overline{b_k}(se^{-r\tau})\, g_k(\tau)\, d\tau \tag{30}$$

Equations (28), (29) and (30) comprise the redefined recursive relationships for the net present worth in terms of the Laplace transforms of the density functions. In order to further simplify the calculations, Perrakis and Henin apply the Taylor's series expansion to yield the following analytical expressions for the moments of the present worth of any cash flow path:

$$m_j = \text{j-th moment about the origin}$$

$$= (-1)^j \, \partial^j \overline{b_k}(s)/\partial s^j \Big|_{s=0} = (-1)^j \cdot j! \cdot \Psi_j \tag{31}$$

where

$$\Psi_j = \Psi_{k,j}\ \overline{g_k}(rj) \tag{32}$$

$$\Psi_{i+1,j} = \sum_{m=0}^{j} b_m \Psi_{i,j-m}\ \overline{g_k}[(j-m)r] \tag{33}$$

$$\Psi_{1,j} = b_j\quad . \tag{34}$$

Recall that the moments generated by equations (31) through (34) are about the origin and must be modified to yield moments about the mean.

The practical difficulties with using this approach to solve application problems is illustrated in a paper by Barnes and Zinn [1]. The procedure analytically yields the Laplace transform of the probability distribution of present worth. In a decision tree, a different analytical problem would be solved for each path considered. The Laplace transform contains total information on the distribution; inverting the transform is often beyond reasonable effort, but the transform may be repeatedly differentiated to give all moments. Machine implementation of transform methods, without human analytical mathematics work, seems out of the question.

CHAPTER III

SERIES OF VARIABLE DURATION ACTIVITIES

1. Solution Procedure for a Series of Cash Flows

Expressions for the expected present worth of a series of cash flows occurring at random times are developed in this section. The assumption is made that all cash flow amounts and all activity durations are independent random variables. Single cash flows occur at the start or end of an activity; uniform continuous cash flows occur throughout an activity. Thus, the timings of single cash flows and the start and end times of uniform continuous cash flows are dependent on previous timings, but are described by random variables whose probability distributions are independent of previous timings.

1.1 Series of Single Cash Flows with Random Timing Intervals

The expected present worth of a single cash flow F occurring at time T in the future is given by

$$E\{F\beta^T\} = E\{F\} \cdot E\{\beta^T\} = E\{F\} \cdot E\{e^{-rT}\} \tag{1}$$

where β is the present-worth discount factor as previously defined, $r = -\ln\beta$, and F and T are independent. Young and Contreras [55] have provided expressions for $E(\beta^T)$ for most commonly encountered distributions of T, and they have given an approximation applicable when only the mean and variance of T are known. Table 1 summarizes their results for single cash flows with random timing.

Table 1. Expected Present Worth Discount Factors for Single Randomly-Timed Cash Flows (Adapted from Young and Contreras [55])

Probability Distribution of T	$E\{\beta^T\}=E\{e^{-rT}\}$ where $r=-\ln\beta$
1. Constant $T = N$	β^N
2. Rectangular with bounds a and b $f(T) = \begin{cases} 1/(b-a) & , a \leqq T < b \\ 0 & , \text{otherwise} \end{cases}$	$\dfrac{\beta^a - \beta^b}{r(b-a)}$
3. Negative - Exponential $f(T) = (1/m)e^{-T/m}$	$\dfrac{1}{1 + mr}$
4. Gamma with mean $m = k/\lambda$ and variance $V = k/\lambda^2$ $f(T) = \dfrac{\lambda(\lambda T)^{k-1}e^{-\lambda T}}{\Gamma(k)}$	$(1 + (Vr/m))^{-m^2/V}$
5. Normal with mean m and variance V $f(T) = \dfrac{\exp(-\frac{1}{2}(T-m)^2/V)}{\sqrt{2\pi V}}$	$\exp(-mr + \frac{1}{2}Vr^2)$
6. Arbitrary T with mean m and variance V	$\simeq e^{-rm}(1 + \frac{1}{2}Vr^2)$

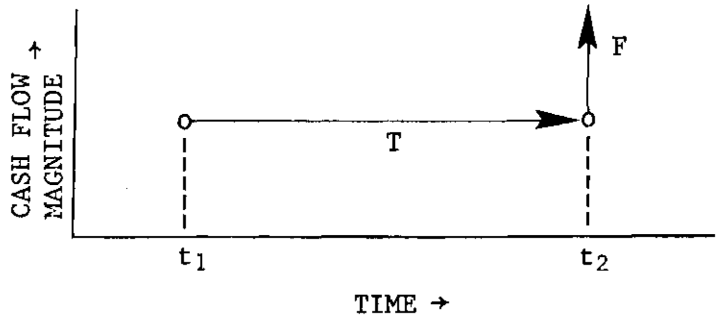


Figure 2. Cash Flow F Occurring
at the Random Time t_1

Let the epochs of a cash flow series be denoted by $\eta = 0,1,2,\ldots,k$, and let an epoch occur upon payment of a single cash flow or upon cessation of a uniform series. Let the times of these epochs be denoted by t_η, and let the epochs be sequential in time so that $t_{\eta+1} > t_\eta$. Let F_η denote the single cash flow that occurs at time t_η. To avoid writing the expectation operator repeatedly, let μ_η denote the expected value of F_η and let α_η denote the expected value of β^{T_η}:

$$\mu_\eta = E(F_\eta) \tag{2}$$

and

$$\alpha_\eta = E(\beta^{T_\eta}) \quad . \tag{3}$$

Let T_i denote the time interval between epoch i-1 and epoch i:

$$T_i = t_i - t_{i-1} \quad . \tag{4}$$

Throughout this work we use the upper-case symbol T to denote a time <u>interval</u> (such as the time between two successive payments or the duration of a payment series) and the lower-case symbol t to denote time relative to a fixed zero. We define $t_0 = 0$ and $T_0 = 0$, so that the initial epoch $\eta = 0$ serves to give a zero to the time scale. The initial payment F_0 occurs at time $t_0 = 0$ or epoch 0, and the final payment F_k occurs at time t_k or epoch k.

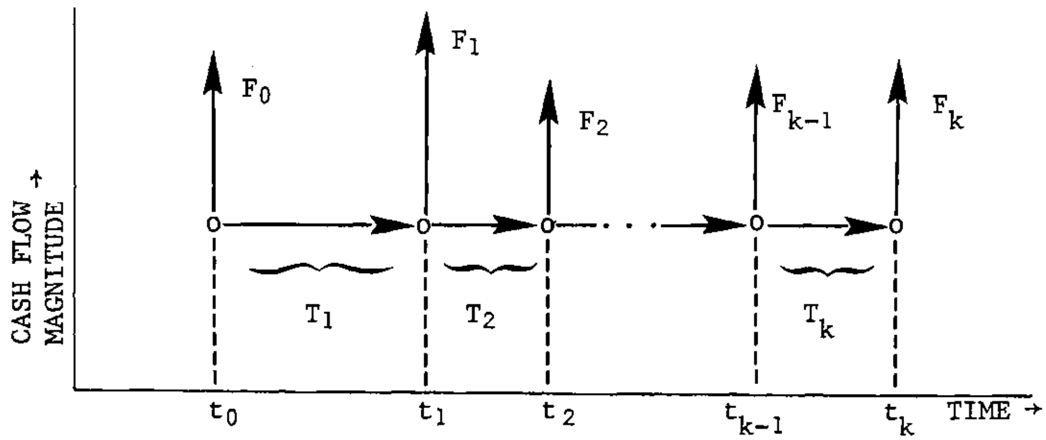


Figure 3. Series of Cash Flows Occurring at Random Times

Let P_i denote the present worth of the series of single payments $F_i, F_{i+1}, \ldots, F_k$. That is, the present worth of those payments that occur at times t_i and later:

$$P_i = \sum_{\eta=i}^{k} F_\eta \beta^{T_\eta} \tag{5}$$

Of course, P_0 is the present worth of the entire series. Let S_i denote the current worth of the same series of payments, defined as follows:

$$S_i = P_i \beta^{-t_i} = \sum_{\eta=i}^{k} F_\eta \beta^{(t_\eta - t_i)} \quad , \ i=0,1,\ldots,k \tag{6}$$

S_i is called a 'current worth' because it is numerically equal to the present worth of the same series of payments as of the time of the first of the payments; that is, a decision maker at time t_i would calculate a present worth of the series of payments occurring at times $t_i, t_{i+1}, \ldots, t_k$ as S_i.

For any realization of the timings $t_1, t_2, \ldots, t_k$, the present worth P_0 may be calculated recursively. From (5) at $i = 0$

$$P_0 = S_0 \tag{7}$$

Now, each current worth S_i can be calculated from the next current worth S_{i+1}, for by (5)

$$S_i = \sum_{\eta = i}^{k} F_\eta \beta^{(t_\eta - t_i)} \tag{8}$$

$$= F_i + \sum_{\eta = i+1}^{k} F_\eta \beta^{(t_\eta - t_i)} \tag{9}$$

$$= F_i + \sum_{\eta = i+1}^{k} F_\eta \beta^{(t_\eta - t_{i+1})} \beta^{(t_{i+1} - t_i)} \tag{10}$$

$$= F_i + \beta^{T_{i+1}} S_{i+1} \tag{11}$$

Since the cash flow series ends at t_k, $S_k = F_k$. Thus, the following series of recursive equations gives P_0:

$$S_{k-1} = F_{k-1} + \beta^{T_k} F_k \qquad (12)$$

$$S_{k-2} = F_{k-2} + \beta^{T_{k-1}} S_{k-1} \qquad (13)$$

$$\vdots$$

$$P_0 = S_0 = F_0 + \beta^{T_1} S_1 \qquad (14)$$

Furthermore, the expected present worth of the entire series, $E(P_0)$, can be calculated recursively. From (2), (3), and (11), we take the expected value of equations (12) through (14), noting that T_η and S_η are independent.

$$E(S_{k-1}) = \mu_{k-1} + \alpha_k E(S_k) = \mu_{k-1} + \alpha_k \mu_k \qquad (15)$$

$$E(S_{k-2}) = \mu_{k-2} + \alpha_{k-1} E(S_{k-1}) \qquad (16)$$

$$\vdots$$

$$E(P_0) = E(S_0) = \mu_0 + \alpha_1 E(S_1) \qquad (17)$$

which proves the following lemma:

<u>The expected present worth, at discount factor β, of a series of cash flows $F_0, F_1, \ldots, F_k$, occurring at times $0, t_1, \ldots, t_k$, respectively, where the time intervals $T_i = t_i - t_{i-1}$ between successive cash flows are random variables independent of each other and of the cash flow amounts, is given by:</u>

$$E(P_0) = \sum_{\eta=0}^{k} \left[\mu_\eta \prod_{i=0}^{\eta} \alpha_i \right] \tag{18}$$

where $\mu_\eta = E(F_\eta)$ and $\alpha_i = E(\beta^{T_i})$.

The proof can be made explicit by expanding (18) and the group of equations (15), (16), and (17); the two expansions are identical term-by-term (recall that $\alpha_0 = 1$).

1.2 Series of Continuous Uniform Cash Flows with Random Time Intervals

The present worth of a uniform continuous cash flow, $\bar{A}$, with duration T is $\bar{A}(1-\beta^T)/r$. Where T is a random variable conforming to the independence assumptions presented earlier, the expected present worth is given by:

$$E\{\bar{A}(1-\beta^T)/r\} = E\{\bar{A}\} \cdot (1 - E\{\beta^T\})/r \tag{19}$$

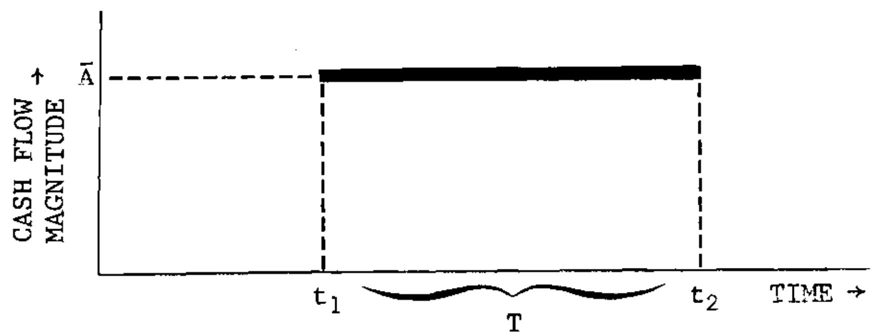


Figure 4. A Continuous Cash Flow $\bar{A}$ with Duration T

The insertion of expressions for $E\{\beta^T\}$ provided by Young and Contreras [55] yields closed-form expressions for $(1 - E\{\beta^T\})/r$ which are summarized in the following table for five commonly encountered distributions

Table 2. Expected Present Worth Discount Factors for Continuous Uniform Randomly-Timed Cash Flows (Adapted from Young and Contreras [55])

Probability Distribution of T	$E\{\beta^T\}=\bar{A}(1-E\{\beta^T\})/r$, where $r=-\ln\beta$
1. Constant $T = N$	$\bar{A}(1-\beta^N)/r$
2. Rectangular with bounds a and b $f(T) = \begin{cases} 1/(b-a) & , A \leqq T < b \\ 0 & , \text{otherwise} \end{cases}$	$\bar{A}\left[\frac{1}{r} - \frac{(\beta^a - \beta^b)}{r^2(b-a)}\right]$
3. Negative - Exponential $f(T) = (1/m)e^{-T/m}$	$\frac{\bar{A}m}{(1 + mr)}$
4. Gamma with mean $m = k/\lambda$ and variance $V = k/\lambda^2$ $f(T) = \lambda(\lambda T)^{k-1}e^{\lambda T}/\Gamma(k)$	$\frac{\bar{A}}{r}\left[1 - (1 + Vr/m)^{-m^2/V}\right]$
5. Normal with mean m and variance V $f(T) = \frac{\exp(-\frac{1}{2}(T-m)^2/V}{\sqrt{2\pi V}}$	$\frac{\bar{A}}{r}\left[1 - e^{(-mr + \frac{1}{2}Vr^2)}\right]$
6. Arbitrary T with mean m and variance V	$\simeq \frac{\bar{A}}{r}\left[1 - e^{-rm}(1 + \frac{1}{2}Vr^2)\right]$

of T and also for the approximation requiring knowledge of only the mean and variance of the distribution for T.

Let

$$\omega_\eta = E\{\bar{A}_\eta\} \tag{20}$$

and

$$\nu_\eta = (1 - E\{\beta^{T_\eta}\})/r = (1-\alpha_\eta)/r \tag{21}$$

where η is the indicator used to associate ω and ν terms with corresponding epochs. An epoch occurs upon the cessation of a uniform continuous cash flow.

Recall that S_i is the current worth at epoch i and, as such, is the time t_i worth of all cash flows occurring within epochs γ, such that $i \leqq \gamma \leqq k$. For the continuous uniform cash flow series, S_i is defined as:

$$S_i = P_i \beta^{-t_i}$$

$$= \sum_{\eta=i}^{k} \bar{A}_{\eta+1} \left[\frac{1 - \beta^{(t_{\eta+1} - t_\eta)}}{r} \right] \beta^{(t_\eta - t_i)} \quad , \tag{23}$$

$$i = 0,1,\ldots,k \quad .$$

As in section 1.1,

$$P_0 = S_0 \tag{24}$$

and the determination of S_0 may be made recursively, where each S_i is a function of S_{i+1}. By (23)

$$S_i = \sum_{\eta=i}^{k} \bar{A}_{\eta+1} \left[\frac{1-\beta^{(t_{\eta+1}-t_\eta)}}{-ln\beta} \right] \beta^{(t_\eta - t_i)} \tag{25}$$

$$= \bar{A}_{i+1} \left[\frac{1-\beta^{(t_{i+1}-t_i)}}{-ln\beta} \right] \beta^0$$

$$+ \sum_{\eta=i+1}^{k} \bar{A}_{\eta+1} \left[\frac{1-\beta^{(t_{\eta+1}-t_\eta)}}{-ln\beta} \right] \beta^{(t_\eta - t_i)} \tag{26}$$

$$= \bar{A}_{i+1} \left[\frac{1-\beta^{(t_{i+1}-t_i)}}{-ln\beta} \right]$$

$$+ \sum_{\eta=i+1}^{k} \bar{A}_{\eta+1} \left[\frac{1-\beta^{(t_{\eta+1}-t_\eta)}}{-ln\beta} \right] \beta^{(t_\eta - t_{i+1})} \beta^{(t_{i+1} - t_i)} \tag{27}$$

$$= \bar{A}_{i+1} \left[\frac{1-\beta^{T_{i+1}}}{-ln\beta} \right] + S_{i+1} \beta^{T_{i+1}} \tag{28}$$

Since an epoch marks the end of a uniform cash flow, the last epoch, $\eta = k$, occurring at time t_k, has a current worth of zero:

$$S_k = 0 \tag{29}$$

The following recursion yields P_0:

$$S_{k-1} = \bar{A}_k \left[\frac{1-\beta^{T_k}}{-ln\beta} \right] \tag{30}$$

$$S_{k-2} = \bar{A}_{k-1} \left[\frac{1-\beta^{T_{k-1}}}{-ln\beta} \right] + S_{k-1} \beta^{T_{k-1}} \tag{31}$$

$$\vdots$$

$$P_0 = S_0 = \bar{A}_1 \frac{1-\beta^{T_1}}{-\ln\beta} + S_1\beta^{T_1} \tag{32}$$

The expected present worth of the continuous uniform cash flow series follows from (20), (21), (28), and (29):

$$E\{S_k\} = 0 \tag{33}$$

$$E\{S_{k-1}\} = \omega_k \nu_k \tag{34}$$

$$E\{S_{k-2}\} = \omega_{k-1}\nu_{k-1} + \alpha_{k-1}E\{S_{k-1}\} \tag{35}$$

$$\vdots$$

$$E\{P_0\} = E\{S_0\} = \omega_1\nu_1 + \alpha_1 E\{S_1\} \tag{36}$$

1.3 Combined Single and Continuous Uniform Cash Flows

In this section we will investigate a cash flow path composed of both single and uniform continuous series cash flows. In the prior two sections, a recursive relationship for solving the path was derived for each type of cash flow separately. The results were:

1. Single Cash Flows

$$E\{S_k\} = \mu_k \tag{37}$$

$$E\{S_\eta\} = \mu_\eta + E\{S_{\eta+1}\} \cdot \alpha_{\eta+1} \tag{38}$$

for $\eta = k-1, k-2, \ldots, 1, 0$

2. Continuous Uniform Cash Flows

$$E\{S_k\} = 0 \tag{39}$$

$$E\{S_\eta\} = \omega_{\eta+1}\nu_{\eta+1} + E\{S_{\eta+1}\} \cdot \alpha_{\eta+1} \tag{40}$$

$$\text{for } \eta = k-1, k-2, \ldots, 0, 1$$

where k is the last node in the path. Notice that the right-most term in each current worth formula is $E\{S_{\eta+1}\} \cdot \alpha_{\eta+1}$. This signifies that the only carry-over effect is that of discounting the current worth at epoch $\eta+1$ to its new value as part of the current worth at epoch η. The desired equation is therefore:

$$E\{S_k\} = \mu_k \tag{41}$$

$$E\{S_\eta\} = \mu_\eta + \omega_{\eta+1}\nu_{\eta+1} + E\{S_{\eta+1}\} \cdot \alpha_{\eta+1} \tag{42}$$

$$\text{for } \eta = k-1, k-2, \ldots, 1, 0$$

2. Variance Analysis

The determination of the expected present worth of a series of cash flows is useful for choosing among various alternatives by indicating which decision paths offer the greatest expected present worth. The variance of the present worth is also of interest [21,23,24,43] indicating variability of present worth. This section is concerned with the determination of this variance.

For a single cash flow occurring at the random time T in the future, the variance of the expected present worth, denoted by $\mathrm{Var}\{F\beta^T\}$, was shown to be equal to the following expression by Rosenthal [43].

$$\mathrm{Var}\{F\beta^T\} = (\mu^2+\sigma^2)\ E\{\beta^{2T}\} - \mu^2(E\{\beta^T\})^2 \qquad (43)$$

where μ = expected value of the random variable F

σ^2 = variance for the distribution of F

β = discount factor

T = random variable for the elapsed time before the occurrence of the cash flow.

For a series of single cash flows, let Q_η equal the variance of the current worth at time t_η for all cash flows that occur at time $\gamma \geqq t_\eta$.

At epoch $\eta = k$, time t_k:

$$Q_k = \sigma_k^2 \qquad (44)$$

Recall from equation (3) above that $\alpha_\eta = E\{\beta^T\}$ and, furthermore, let us define:

$$\psi_\eta = E\{\beta^{2T_\eta}\} \quad . \qquad (45)$$

Then, at epoch $\eta = k-1$, time t_{k-1}, application of equation (43) yields:

$$Q_{k-1} = \sigma_{k-1}^2 + (S_k^2 + Q_k)\psi_k - S_k^2\alpha_k^2 \qquad (46)$$

where σ_{k-1}^2 = contribution to variance of the cash flow at time t_{k-1}

S_k = expected present worth at time t_k

Q_k = variance of expected present worth at time t_k

Repetitive application of equation (43) in this manner, where we consider each back-step along the branch separately, will yield the solution for the variance of the branch when we reach Q_θ. The recursive relationship may therefore be defined as follows:

$$Q_k = \sigma_k^2 \tag{47}$$

$$Q_\eta = \sigma_\eta^2 + (S_{\eta+1}^2 + Q_{\eta+1}) \cdot \psi_{\eta+1} - S_{\eta+1}^2 \, \alpha^2{}_{\eta+1} \tag{48}$$

$$\text{for } \eta = k-1, k-2, \ldots, 0, 1$$

It can further be shown that for a uniform continuous cash flow occurring between times t_η and $t_{\eta+1}$ for a duration defined by the random variable T_η, the variance at time t_η is given by:

$$Q_\eta = (\zeta^2 + \omega^2) \cdot (\psi - \alpha^2)/r^2 + \zeta^2 \nu^2 \tag{49}$$

where $\zeta^2 = \text{Var}\{\bar{A}\}$

$\omega = E\{\bar{A}\}$

Since all of the random variables associated with cash flows and timing throughout the model are mutually independent, the variance contributions are additive. Figure 5 represents this situation. The following recursive relationship is hence defined for arcs with both single termination cash flows and continuous uniform cash flows.

$$Q_k = \sigma_k^2 \tag{50}$$

$$Q_\eta = \sigma_\eta^2 + (S_{\eta+1}^2 + Q_{\eta+1})\psi_{\eta+1} - S_{\eta+1}\,\alpha_{\eta+1}^2 \tag{51}$$

$$+ (\zeta_{\eta+1}^2 + \omega_{\eta+1}^2) \cdot (\psi_{\eta+1} - \alpha_{\eta+1}^2)/r^2$$

$$+ \zeta_{\eta+1}^2\,\nu_{\eta+1}^2$$

$$\text{for } \eta = k-1, k-2, \ldots, 1, 0 \quad .$$

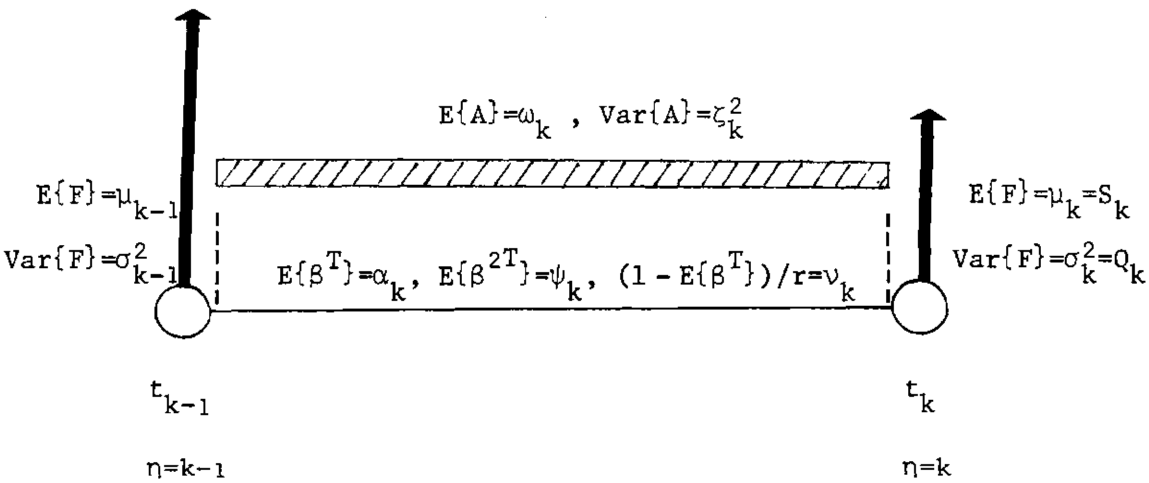


Figure 5. Graphical Representation of Variance of Cash Flows Associated with a Single Activity

3. Summary

In this chapter, expressions for the expected present worth of a series of cash flows are derived. Equation (42) allows a "value" to be assigned to every node of the decision tree during the foldback (dynamic programming) solution of a generalized decision tree. This "value" is the expected present worth, at the time of the epoch represented by the node, of all cash flows from that epoch onward. Equation (51) allows the variance of the present worth to be calculated for every node. Thus, a generalized decision tree may be solved in the same manner as a conventional decision tree. Equations (42) and (51) require the probability distribution of every activity duration; whereas conventional decision trees require fixed durations.

In Chapter IV, the Young and Contreras approximation to solve generalized trees, where only the first two moments of the activity durations are given, is tested.

CHAPTER IV

VALIDATION OF THE SERIES APPROXIMATION METHOD

This chapter is concerned with identifying the error in using an approximation formula for the expectation of present worth.

If only the mean, m, and the variance, V, of an activity duration, T, are known, rather than the entire probability distribution of T, expected present worths have been shown to be accurately estimated by use of an approximation for $E\{\beta^T\}$ [55]:

$$E\{\beta^T\} \simeq \beta^m(1 + \tfrac{1}{2}Vr^2) \tag{1}$$

This approximation is based on a Taylor series. The employment of equation (1) to obtain approximations of expected present worths shall henceforth be called the "series approximation method" (SAM).

Although the approximation for $E\{\beta^T\}$ is known to be relatively accurate, it has not previously been determined whether the expected present worth of a <u>series</u> of independent activities is well approximated by using SAM for each of the activities in the series. A path in a decision tree consists of a series of activities, and systematic error could build up due to the fact that the discounting of a cash flow depends on the durations of all preceding activities. Thus it is of interest to study the accuracy of the SAM approximation of products of approximated discount factors such as $\beta^{T_{oi}}\, \beta^{T_{il}} \ldots \beta^{T_{jk}}$ where $T_{oi} + T_{il} + \ldots T_{jk}$ is the time to node k from node o.

The analysis is divided into two sections; one concerned with the error propogation along discounted cash flow paths, and the second with identifying the "cost" of this error as it relates to the evaluation of generalized decision trees. The "cost" is a function of both the probability that an incorrect decision path will be selected and the difference between the expected present worth of the selected sub-optimal path and that of the correct optimal path.

For the determination of the expected present worth of the cash flows associated with an arc, where the arc duration is a random variable, two characteristics are significant; the discount factor and the shape of the probability density function for the time duration. The discount factor is given by β, and the shape of the density function is partially characterized by the mean, m, and the variance, V. The coefficient of variation, $c = \sqrt{V}/m$, is used as an indicator of the shape of the distribution. This analysis will therefore be concerned with the error as it relates to various values for c and β.

1. Single Branch Series Cash Flow Analysis

This section deals with the error propogated along cash flow paths. Two cases are analyzed. First, the case where the activity duration densities are equal for each arc in the path is considered in order to establish general limitations for the characterizing parameters c and β. This condition is included in the analysis because a closed form equation for the solution is available. Second, paths with randomly generated activity duration densities are studied. These types of paths are selected in order to demonstrate the error propogation for cash flow paths that would typically be encountered in practice.

For this study, the cash flow amounts are set equal to 1.

A recent study conducted by Fremgen [14] shows that 88% of businesses and 86% of military installations use a planning horizon of five years or less for planning capital expenditures. The table below shows the results of the survey which was conducted with 177 business firms and 70 military installations. In accordance with these findings, a mean path duration of five years is used.

Table 3. Maximum Number of Years for which Capital Expenditures are Planned

Years	Percentage of Business Respondents	Percentage of Military Respondents
2	12	23
3	21	8
4	0	3
5	55	52
> 5	11	14
	99	100

1.1 Series with Equal Activity Duration Densities and Equivalent Payments

For a series of n equal cash flows where the timing between each cash flow is independent and identically distributed it can be shown that:

$$E\{P_0\} = F\alpha[(1-\alpha^n)/(1-\alpha)] \tag{2}$$

for the case where there is no cash flow at time 0, and:

$$E\{P_0\} = F[(1-\alpha^n)/(1-\alpha)] \tag{3}$$

for the case that includes a cash flow at time 0. For equations (2) and (3):

$$\alpha = E\{e^{-rT}\} \quad \text{and} \quad F = \text{Constant}.$$

Equation (2) was used to generate error statistics for cash flow paths with from 1 to 10 activities each with an average duration of one year. The interest rate was varied between 1% to 60% in increments of 0.1%. This was intended to demonstrate an upper limit for the interest rate for a given percentage error in the net present worth.

Figure 6. Path with Equal Activity Duration Densities

The error was determined as the difference between the resultant net present worth using the series approximation formula and the exact formula for α, given that T is gamma distributed. The gamma distribution was used because it is able to take on a wide variety of shapes.

Error statistics were generated for a range of gamma functions depicted by the coefficient of variation. Since the mean duration was fixed at the value 1, this measure of dispersion was altered by varying the variance of the gamma function. Distributions included in the experiment ranged from bell-shaped functions with $c < 1.0$, to the skewed exponential distribution with $c = 1.0$, to the highly skewed functions where $c > 1.0$. Table 4 summarizes the results of the test.

Table 4. Error Statistics for Identically Distributed Cash Flow Paths

Coefficent of Variation	Variance of Gamma Dist.	No. of Activities in the Branch	Highest Interest Rate Yielding Specified % Error or Less <0.1%	<1.0%	<5.0%	Maximum Percentage Error Encountered
0.500	0.25	1	48.2	60.0+	60.0+	< 1.0%
		5	32.9	60.0+	60.0+	< 1.0%
		10	28.1	60.0+	60.0+	< 1.0%
0.707	0.50	1	27.6	60.0+	60.0+	< 1.0%
		5	18.8	51.9	60.0+	< 5.0%
		10	15.6	45.0	60.0+	< 5.0%
1.000	1.00	1	16.5	42.7	60.0+	< 5.0%
		5	11.2	28.2	60.0+	< 5.0%
		10	9.2	23.5	53.2	6.1%
1.500	2.25	1	9.3	22.9	48.1	7.5%
		5	6.3	15.1	30.1	19.5%
		10	5.1	12.3	24.6	28.2%

For c = 1.5, a gamma function with a coefficient of skewness equal to 3.0, the Series Approximation Method yields an error of less than 1% for interest rates up to 15.1% for a duration of five years. When c is lowered to a value of 1.0 the maximum acceptable interest rate of 28.2% is allowed for an error of less than 1%. This implies that an upper limit for the interest rate may be in the neighborhood of 25% if a 30% limitation on acceptable values for the coefficient of variation is also applied. This will be investigated more fully later.

1.2 Series with Randomly Generated Activity Durations and Equal Cash Flows

Typical cash flow streams involve receipts and disbursements that occur at various epochs in the future. The times between epochs are usually not identically distributed as was assumed in the previous section. This section is devoted to the determination of the error associated with using the series approximation formula to describe the time duration between epochs where the duration densities are not equal.

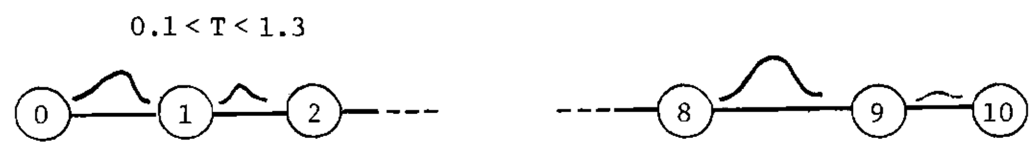


Figure 7. Path with Random Activity Duration Densities

The procedure selects an interest rate and randomly generates timing distributions, based on the mean and variance characterizing the distribution, for branches composed of 10 nodes. In this way, it is intended to simulate the occurrence of cash flow streams which would typically be encountered in practice. It is assumed that the actual distribution of the timings is gamma and that the coefficient of variation is never greater than $c = 1$. 1000 paths were generated with a mean duration between epochs of 0.7 year and a range of from 0.1 year to 1.3 years. The standard deviation of the 10,000 epoch average time durations generated is 0.4 year. Tables 5, 6 and 7 summarize the results of the test for interest rates ranging from 12% to 60%.

The test was broken into three parts. The first part was concerned with the error propagation along paths with only termination cash flows, the second part investigated paths with only continuous uniform cash flows, and the third part investigated the joint case of both continuous cash flows and termination cash flows.

The first test concerning only termination returns resulted in an error of less than 0.7% between the net present worth computed by the series approximation formula and the desired result based on the gamma distribution for an interest rate of 36%. As the interest rate was increased to 60% an error of 1.3% was encountered for a branch with nine activities. 944 of the 1000 branches generated with nine activities at an interest rate of 60% yielded an error of less than 1%.

The second test involving only continuous uniform cash flows resulted in all generated branches having an error of less than 1%. The maximum encountered error was largest for the smallest interest rate of 12% and amounted to an error of 0.5%.

The third test, intended to demonstrate the most general case with both termination cash flows and continuous uniform cash flows resulted in errors less than 0.3% for interest rates below 36%. The highest percentage

Table 5. Simulated Series of Single Cash Flows

Interest Rate	No. of Activities in the Branch	No. of Cases with Specified Error <1.0%	<5.0%,>1.0%	>5.0%	Maximum Percentage Error Encountered
12.0%	1	1000	0	0	< 0.1%
	5	1000	0	0	< 0.1%
	9	1000	0	0	0.1%
24.0%	1	1000	0	0	0.1%
	5	1000	0	0	0.2%
	9	1000	0	0	0.3%
36.0%	1	1000	0	0	0.1%
	5	1000	0	0	0.5%
	9	1000	0	0	0.7%
48.0%	1	1000	0	0	0.2%
	5	1000	0	0	0.8%
	9	999	1	0	1.0%
60.0%	1	1000	0	0	0.4%
	5	975	25	0	1.1%
	9	944	56	0	1.3%

Table 6. Simulated Series of Continuous Uniform Cash Flows

Interest Rate	No. of Activities in the Branch	No. of Cases with Specified Error <1.0%	<5.0%,>1.0%	>5.0%	Maximum Percentage Error Encountered
12.0%	1	1000	0	0	0.2%
	5	1000	0	0	0.4%
	9	1000	0	0	0.5%
24.0%	1	1000	0	0	0.2%
	5	1000	0	0	0.2%
	9	1000	0	0	0.2%
36.0%	1	1000	0	0	0.3%
	5	1000	0	0	0.2%
	9	1000	0	0	0.1%
48.0%	1	1000	0	0	0.4%
	5	1000	0	0	0.2%
	9	1000	0	0	0.1%
60.0%	1	1000	0	0	0.4%
	5	1000	0	0	0.2%
	9	1000	0	0	0.1%

Table 7. Simulated Series of Combined Single and Continuous Uniform Series Cash Flows

Interest Rate	No. of Activities in the Branch	No. of Cases with Specified Error <1.0%	<5.0%,>1.0%	>5.0%	Maximum Percentage Error Encountered
12.0%	1	1000	0	0	0.1%
	5	1000	0	0	0.1%
	9	1000	0	0	< 0.1%
24.0%	1	1000	0	0	0.1%
	5	1000	0	0	0.1%
	9	1000	0	0	0.1%
36.0%	1	1000	0	0	0.2%
	5	1000	0	0	0.1%
	9	1000	0	0	0.3%
48.0%	1	1000	0	0	0.2%
	5	1000	0	0	0.3%
	9	1000	0	0	0.5%
60.0%	1	1000	0	0	0.3%
	5	1000	0	0	0.4%
	9	1000	0	0	0.7%

error encountered was 0.7% at an interest rate of 60% with nine activities.

To place these results in economic perspective, one can arbitrarily choose "practical" limits for interest rate and coefficient of variation. For example, present worths are normally calculated at an interest rate equal to the rate of return normally earned in the ordinary course of business (the "minimum attractive rate of return"), which is less than 24% for most business concerns in capital economies,* and probability distributions with coefficients of variation greater than $c = 1$ are rarely encountered.

With $i \leq 20\%$ and $c \leq 1$, the tests imply that an error of less than 0.5% is to be expected through the use of SAM. Since it is unlikely that the numbers used in a decision tree study would be significant beyond three digits, an error of 0.5% or less would impact only the least significant digit. It is therefore appropriate to conclude that the use of the series approximation formula will yield appropriate results in practical situations.

2. Generalized Decision Trees

In section IV.1, the error incurred by the use of the series approximation method (SAM) as an estimator of $E\{\beta^T\}$ was shown to be valid for $i < 0.24$ and $c < 1.0$ for series of cash flows where the problem formulation was accurate to three significant digits. This section is concerned

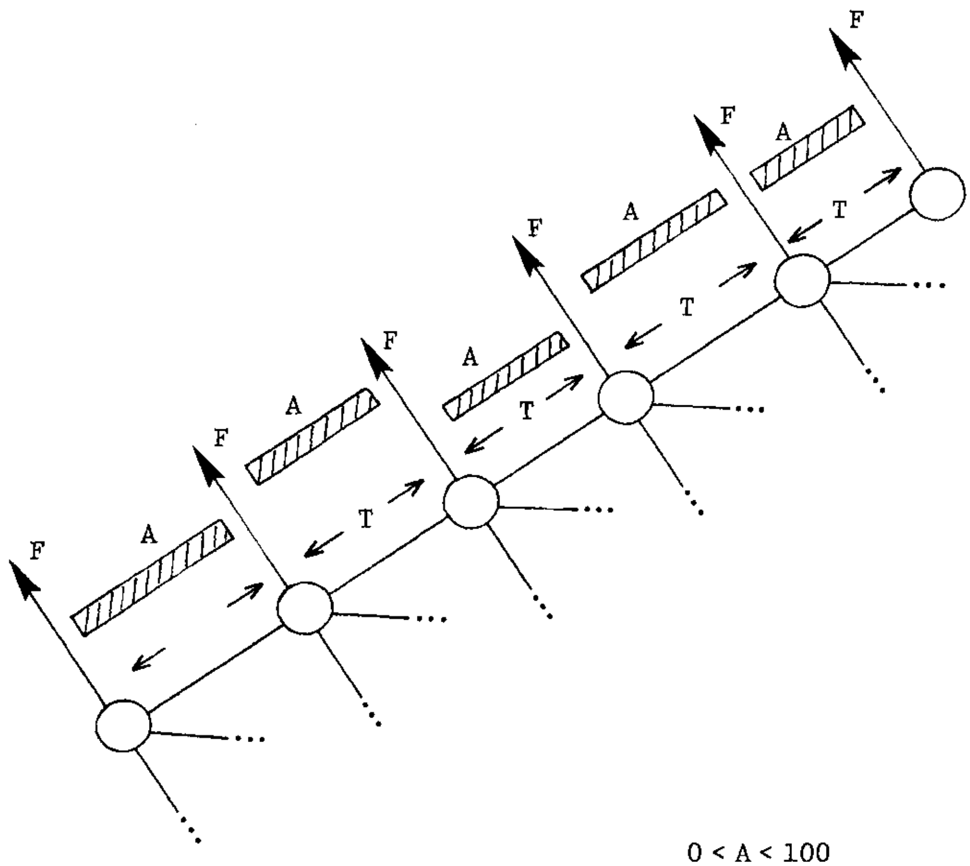
*Formal uses of interest rates higher than 20% are in fact in common use, but as compensation for modeling deficiencies assumed not to exist among users of generalized decision trees. These deficiencies include systematic bias in overestimating revenues and underestimating costs, infinite-life modeling of finite-life revenues, estimation of future cash flows in nominal rather than inflation-free dollars, safety-factor conservatism, and lack of explicit consideration of risk and uncertainty in deterministic methods.

with the impact that this estimator will have on the solution of a decision tree. Three types of decision trees are of interest; trees with only termination cash flows, trees with only continuous uniform cash flows, and trees that jointly combine the two types of returns. This will cover all possible user cases.

For this test, a program was prepared that randomly generates a decision tree with 363 arcs and 364 nodes. The tree has five epochs. Three arcs extend from each node. The mean duration of each epoch is between 0.1 year and 2.0 years. The typical arc has a mean duration of 1.01 year. The variance of the time duration is determined by the selected coefficient of variation. The paths thus generated within the tree are not intended to represent typical flows to be found in practice as in Section IV.1.2; here it is intended that the worst case will be revealed. For each selection of the parameters c and i, 100 trees are randomly generated. Each tree is evaluated using both the SAM estimator and the gamma function for $E\{\beta^T\}$. The correct time distributions are assumed to be gamma to provide a basis for evaluating the SAM solutions.

In accordance with the results of Section IV.1.2, the interest rate is varied between 0.06 and 0.30. The coefficient of variation is tested for the three values in Section IV.1.1, namely 0.5, 1.0 and 1.5. These parameter selections are intended to demonstrate results for all trees of practical interest plus a representative sample of trees that extend beyond reasonable limits. The case where $c = 1.0$ and $i = 0.24$ is intended to represent the reasonable limit.

The results of the test are summarized in Tables 8 through 10. Table 8 indicates the average percentage error between the SAM solution



$0 < A < 100$

$0 < F < 100$

$0.1 < T < 2.0$

Figure 8. Representation of Tree Structure for Validation Test

Table 8. Average of the Percentage Error Between Series Approximation Method and Gamma Evaluated Expected Present Worths of Decision Trees

	$0 < F < 100, A = 0$			$F = 0, 0 < A < 100$			$0 < F < 100, 0 < A < 100$		
c / i	0.50	1.00	1.50	0.50	1.00	1.50	0.50	1.00	1.50
0.06	0.00	0.00	0.02	0.02	0.27	1.32	0.02	0.17	0.81
0.12	0.00	0.03	0.13	0.05	0.76	3.83	0.03	0.41	1.92
0.18	0.01	0.08	0.36	0.08	1.26	6.52	0.04	0.55	2.60
0.24	0.01	0.14	0.71	0.10	1.66	9.03	0.04	0.59	2.73
0.30	0.01	0.20	1.11	0.12	2.04	11.44	0.04	0.61	2.53

Table 9. Number of Times an Incorrect Decision Path was Selected by the Series Approximation Method

	$0 < F < 100, A = 0$			$F = 0, 0 < A < 100$			$0 < F < 100, 0 < A < 100$		
i \ c	0.50	1.00	1.50	0.50	1.00	1.50	0.50	1.00	1.50
0.06	0	0	0	0	0	0	0	0	0
0.12	0	0	0	0	0	1	0	0	6
0.18	0	0	0	0	1	5	0	1	16
0.24	0	0	10	1	1	5	0	3	12
0.30	0	5	6	0	5	5	0	5	29

Table 10. Maximum Percentage Error Between the Expected Present Worth of SAM and GAMMA Selected Paths

	$0 < F < 100, A = 0$			$F = 0, 0 < A < 100$			$0 < F < 100, 0 < A < 100$		
c / i	0.50	1.00	1.50	0.50	1.00	1.50	0.50	1.00	1.50
0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.12	0.00	0.00	0.00	0.00	0.00	0.23	0.00	0.00	0.09
0.18	0.00	0.00	0.00	0.00	0.26	0.64	0.00	0.00	0.74
0.24	0.00	0.00	0.30	0.02	0.12	0.71	0.00	0.27	1.73
0.30	0.00	0.21	0.93	0.00	0.59	1.52	0.00	0.38	2.98

and the gamma solution for the expected net present worth of the 100 trees. For $c = 1.0$, the limiting case, the average error encountered for the case of termination cash flows only was 0.2%. For the case of continuous uniform cash flows only, the error was significantly higher reaching 1.7% for $i = 0.24$. For the case of the combined cash flow types, the expected error was 0.61% at $i = 0.3$. Hence, for the limiting case, we may expect the worst-case prospect of an expected error of 1.7% where $c = 1.0$ for all duration densities in the tree and only continuous uniform cash flows are considered.

Table 9 indicates the number of times the calculated optimal paths differed. As expected, the number of times the path was incorrectly selected by SAM increased with increasing i and c. Table 10 indicates the cost of the incorrectly selected path. The cost is computed by recalculating the expected net present worth of the SAM selected path with the gamma equation so that the true error may be recorded. For the limiting case, when a wrong decision was made, the maximum percentage error between the new present worth of this path and that of the correct path was 0.27%. This percentage error is well within the boundaries of model accuracy for three significant digits.

In Table 9, numerous incorrect decisions were recorded when in fact the expected present worth of the incorrectly selected path was not significantly different from the correct path selected by the gamma function. In Table 11, only those cases where the cost of the incorrect decision was greater than 0.5% are recorded. Based on the results presented in this table, no significantly wrong decisions were made for any of the models generated up to the limiting values for the parameters c and i.

Table 11. Number of Times an Incorrect Decision Path Was Selected by the Series Approximation Method Where the Percentage Error Between the Expected Present Worth of the SAM and GAMMA Selected Paths is Greater than 0.5%

	$0 < F < 100, A = 0$			$F = 0, 0 < A < 100$			$0 < F < 100, 0 < A < 100$		
i \ c	0.50	1.00	1.50	0.50	1.00	1.50	0.50	1.00	1.50
0.06	0	0	0	0	0	0	0	0	0
0.12	0	0	0	0	0	0	0	0	0
0.18	0	0	0	0	0	4	0	0	3
0.24	0	0	0	0	0	2	0	0	8
0.30	0	0	1	0	1	4	0	0	17

This fact adds further credibility to the SAM method for the limitations of $i \leq 0.24$ and $c \leq 1.0$.

Using the results tabulated in Table 11, Table 12 was constructed to give an indication of the liklihood of making the correct decision given maximum acceptable values for c and i. Based on the maximum values of $c = 1.0$ and $i = 0.24$, the probability of SAM yielding the correct path is greater than or equal to 0.97. For the cases where SAM fails to yield the correct path, the proportional cost of this wrong decision is less than 0.3%. The maximum expected costs for wrong decisions are depicted in Table 13 for corresponding maximum c and i values.

3. Summary

In Section IV.1.1, the analysis showed a marked increase in the error between the SAM calculated present worth and the gamma calculated present worth for $c > 1.0$. For a coefficient of variation equal to 1.0, the error was found to be less than 1% for interest rates up to 28.2% for a path of five years duration, and up to 23.5% for a path of ten years duration. This implies that, for a five-year planning horizon, the approximation is accurate to less than 1% error for interest rates below 28% and densities that are no more skewed than the exponential.

In Section IV.1.2, typical discounted cash flow paths were tested where c was randomly selected between the values of 0 and 1. The net result showed a maximum expected error less than 0.3% for interest rates below 24% where the mean duration of the path was 6.3 years. It is intuitively plausible to represent the duration of a single activity by a density as skew as the exponential density $(c = 1)$. For an arc that

Table 12. Minimum Expectation of a Correct Decision Using the Series Approximation Method by Maximum c and i

Max i \ Max c	0.50	1.00	1.50
0.06	1.00	1.00	1.00
0.12	1.00	1.00	0.94
0.18	1.00	0.99	0.84
0.24	1.00	0.97	0.88
0.30	1.00	0.95	0.71

Table 13. Maximum Expected Error in Incorrect Decision Path by Maximum c and i

Max i \ Max c	0.50	1.00	1.50
0.06	*	*	*
0.12	*	*	0.1%
0.18	*	0.3%	0.8%
0.24	*	0.3%	1.8%
0.30	*	0.6%	3.0%

represents multiple activities, the density for the arc is composed of a series of such distributions, or represented by the gamma distribution with c less than 1. Typically, the minimum attractive rate of return employed by an organization is less than 24%. Hence, we assume practical limitations of $1-(1/\beta) \leq 0.24$ and $c \leq 1.0$. Given these limitations, the series approximation method yields errors less than 0.3%.

Section IV.2 examined the use of SAM to compute optimal decision paths within a decision tree. For decision trees where the values are typically accurate to only three significant digits, the maximum expectation of a wrong decision was found to be 0.03. This was for the case where all $c = 1.0$ and $i = 0.24$. Given that the incorrect path was selected, the maximum error between the SAM and gamma computed expected present values was 0.3%. This implies that the expected cost of a wrong decision, caused by the use of SAM, is less than 0.009%. This is well below the significance of the input variables which suggests that SAM is a sound estimator when used in lieu of the time duration densities along paths in a decision tree.

CHAPTER V

COMPUTER PROGRAM TO SOLVE DECISION TREES WITH STOCHASTIC ACTIVITY DURATIONS

1. Introduction

This chapter describes a FORTRAN-IV computer program that may be used to solve decision trees where the cash flow magnitudes and the time duration between activities are random variables.

Several decision tree programs are currently in existence [36]. Applied Decision Systems, Inc. of Wellesley Hills, Massachusetts developed a program called ADTREE. DuPont had a program that is used internally in batch mode. The IBM Cambridge Scientific Research Center developed a decision tree program that utilizes computer graphics to display the model. This program was developed under an internal experimental program and due to dependency on machines that are no longer produced by IBM is no longer being tested. Scientific Software Corporation of Englewood, California developed a program that uses simulation to solve the model. It is currently in use by the U.S. Department of Defense and the U.S. Geological Survey. Stanford Research Institute had developed two packages. One is an interactive decision tree program called TREE that can handle problems up to 1000 nodes. It is highly system dependent and will handle only a limited number of probability distributions. SRI developed another package called CTREE which may be used for large decision tree problems. CTREE provides macros which must be used in a user-written FORTRAN program to model the problem. Systemes Informatiques De Gestion of Paris, France has a program that is similar to SRI's TREE program and will handle problems with up to 450 nodes. It is used primarily by educational institutions in France.

None of the programs mentioned above provide for variable activity durations.

The computer program described in this chapter will allow for variable activity durations. The distribution of time may be assigned any of five distribution types: constant, rectangular, negative-exponential, normal, and gamma. If the desired distribution type is unknown, the Series Approximation Method, described in Chapter III, may be used to solve the problem. With the use of SAM there is no need to specify the distributions of the random variables beyond the specification of the mean and variance. A discussion of the accuracy of SAM is presented in Chapter IV. The primary limiting factors are that the interest rate specified be less than 0.24 (annual) and that the coefficient of variation for any random variable in the model be less than 1.0. The program will provide warning messages if either of these limitations are violated.

The moments required by the specified distribution type may be entered explicitly; e.g., mean and variance; or in the more intuitive approach of specifying the earliest possible time (LB), the latest possible time (UB), and the most likely time (MODE). By specifying the later parameters, the program will compute the required moments by using the PERT estimation technique as shown below:

$$E\{T\} = (LB + 4\ MODE + UB)/6 \tag{1}$$

$$Var\{T\} = (UB - LB)^2/36 \quad . \tag{2}$$

The following sections describe the data preparation necessary to run the program, a description of the menu flow through the program, the logic used to select the optimal decision paths, and three example problems. The program listing is provided in the Appendix.

1.1 Program Overview

The program is written in standard FORTRAN-IV statements. It consists of a modular design as diagrammed in Figure 9. This is intended to facilitate maintenance and the incorporation of enhancements.

The user interface is controlled by a series of option menus. As depicted by the terminal symbol in Figure 9 there are four primary menus: Initiation, Data Entry and Update, Processing, and Termination. The Initiation Option menu provides access to the Instruction and File Read modules. The Instruction module is itself composed of an option menu which enables the user to obtain assistance based on his scope of interest. The File Read module serves the purpose of retrieving model data from a sequential file. The Data Entry and Update menu provides controlled access to the Data Description modules which provide explicit prompting for the various model parameters (e.g., terminal cash flows, uniform cash flows, time variables). Under this menu prompt, the user may select to either add an arc to the model or change parameters in a currently existing arc. The Processing menu provides access to the Display module which provides the user with the capability to display all model data for arcs from or to a specific node or for all arcs in the model. The Processor menu also controls access to the processor modules which compute the expected present worth and variance of the paths in the tree. The Termination menu allows access to the File Write module which is used to

BASIC PROGRAM FLOW

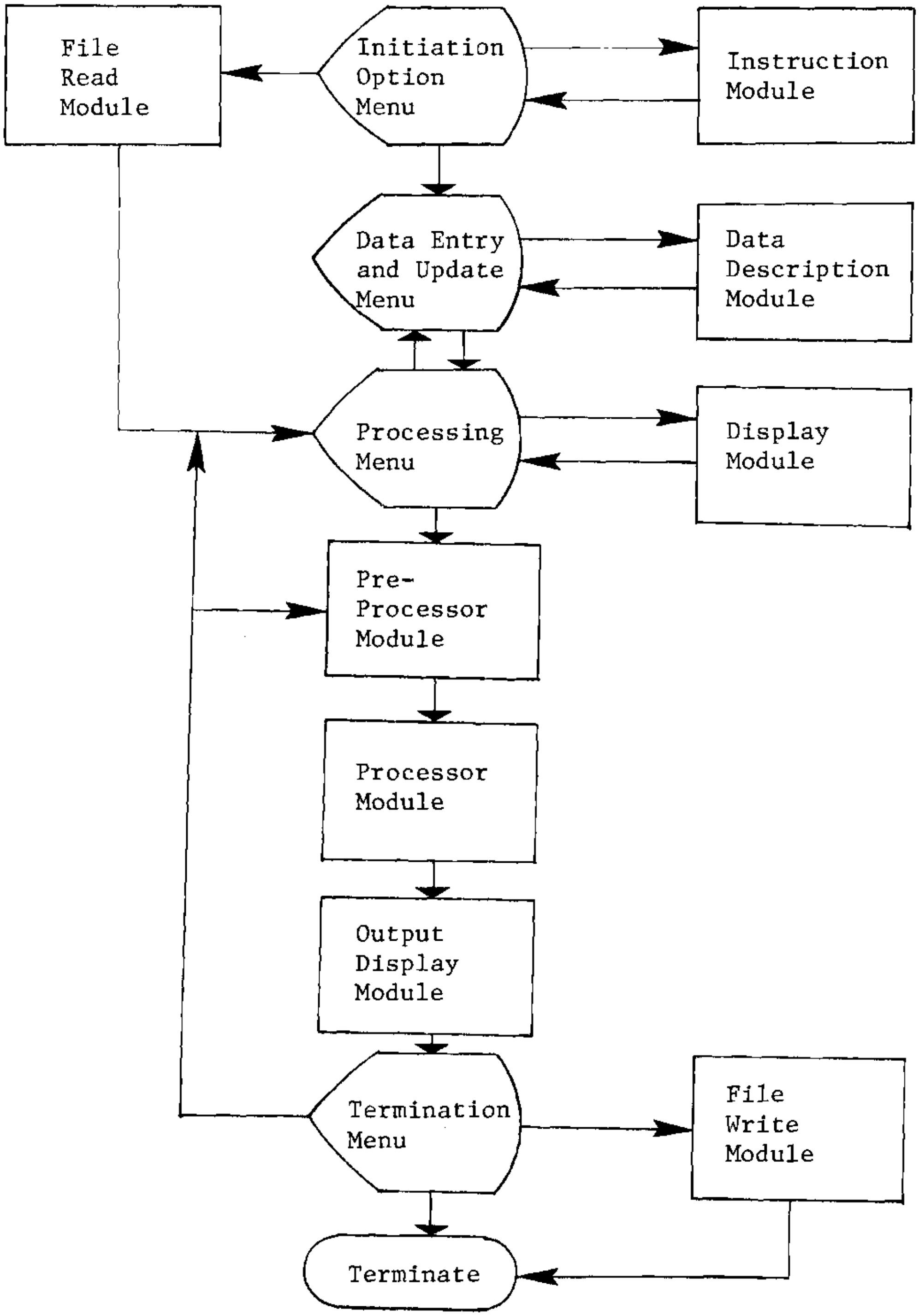


Figure 9. GTREE Program Flow

write the data associated with the current model to a sequential file, and the capability to re-run the model either by branching to the Processing menu or directly to the Pre-processor after specification of a new interest rate. A detailed flowchart of the option processing states is provided in the Appendix.

For user convenience, a worksheet is provided on the following page which may be prepared prior to running the program. The only data entry restriction is that both source node numbers and sink node numbers be numbered sequentially. Samples of prepared worksheets are provided with the example problems later in the chapter.

1.2 Processor Logic

The purpose of the Processor module is to determine the optimal decision paths in the model. This function is divided into two parts; the Pre-processor and the Processor. The Pre-processor transforms the data into a format suitable for the Processor. For the data entry and file read functions, data is maintained in the File Format shown in Figure 10. This ensures the integrity of the user supplied data. For Processor use, all variables described in terms of bounds and the mode are transformed into the corresponding mean and variance. The user may explicitly specify the mean and variance by keying "999" as the first field of each variable class. For the time parameters, the values of α,ν, and ψ are evaluated. The resulting input data to the Processor module is shown in Figure 11. The computations follow from Tables 1 and 2 in Chapter III.

The Processor logic is described by the steps outlined in Figure 12.

WORKSHEET FOR DECISION TREE PROGRAM

ARC IDENTIFICATION		TERMINAL CASH FLOW			CONT. CASH FLOW			TIME DURATION				PROB-ABILITY
Source Node #	Sink Node #	999 LB	Mean UB	Var. MODE	999 LB	Mean UB	Var. MODE	DT	999 LB	Mean UB	Var. MODE	

Definitions: LB = Lower bound
UB = Upper bound
MODE = Most likely value
999 = Specifies that mean and variance will be used.

DT = Distribution type: C = constant
R = rectangular
E = exponential
G = gamma
N = normal
A = arbitrary

DTFILE(I,J=	1	2	3	4	5	6	7	8	9	10	11	12	13
	SOURCE NODE #	SINK NODE #	999 or LB	MEAN or UB	VAR. or MODE	999 or LB	MEAN or UB	VAR. or MODE	DT	999 or LB*	MEAN or UB**	VAR. or MODE	PR
	ADDRESSABILITY		TERMINAL CASH FLOW			UNIFORM CASH FLOW				TIME DURATION			

DT = Distribution type
PR = Probability

*For the constant time duration, the position contains the value.

**For the rectangular time duration, this and the previous position contain the bounds.

Figure 10. Layout of File Data Format

DTWORK(I,J=	1	2	3	4	5	6	7	8	9	10
	SOURCE NODE #	SINK NODE #	MEAN μ	VAR. σ^2	MEAN ω	VAR. ζ^2	α	ν	ψ	PR
	ADDRESSABILITY		TERMINAL CASH FLOW		UNIFORM CASH FLOW			TIME DURATION		PROBABILITY

Figure 11. Layout of Processor Data Format

STEP 1. Begin with the last arc defined in the model.

STEP 2. Determine whether the probability associated with the current arc is less than one. If it is, this signifies that this is a chance arc, go to step 7. Otherwise, go to step 3.

STEP 3. Compute the current worth of this arc. If this value is greater than any current worth previously computed for the source node corresponding to this arc, go to step 4. Otherwise, go to step 5.

STEP 4. Save the current worth of the source node and save the source and sink node numbers representing this desirable path. Go to step 5.

STEP 5. Determine whether the next arc in the model has the same source node number as the arc currently being processed. If it does, go to step 3, otherwise, increment the path counter (to preserve the previously selected path) and go to step 6.

STEP 6. Compute the variance for the selected arc and go to step 2.

STEP 7. Compute the current worth of this arc and multiply by the probability. Go to step 8.

STEP 8. Compute the variance of this arc, multiply by the probability squared and add this value to the current variance value for this source node. Go to step 2.

Figure 12. Processor Logic

2. Example Problems

Three example problems are presented in this section. The first example is a plant construction problem with an option for later expansion based on anticipated market demand. Variable construction times are accomodated. The second example is a bridge construction problem where one of two foundation alternatives must be selected. The variability in the construction time associated with each alternative is critical to this decision problem. The third example is an analysis of a lease versus purchase option for computer equipment where two configurations with different processing capabilities are involved.

2.1 Plant Expansion Example

The first example problem was extracted from a problem presented by Magee [32]. The problem is defined as follows:

> After an extensive market survey, a manufacturer is confronted with the alternatives to either build a large plant facility today, or to build a smaller facility today with an option to expand this plant at a later date.

Figure 13 is a decision tree diagram of the problem and Table 14 provides the detailed data. The numbers have been modified from the original problem posed by Magee to provide for variability in construction time.

The first decision point is represented by node 1. If the manufacturer chooses to build the large plant initially, he will incur a \$750K start-up cost, a charge of \$50K per month while the project is underway, and a termination cost of \$750K. It is expected that construction will take 30 months, however, under ideal conditions construction

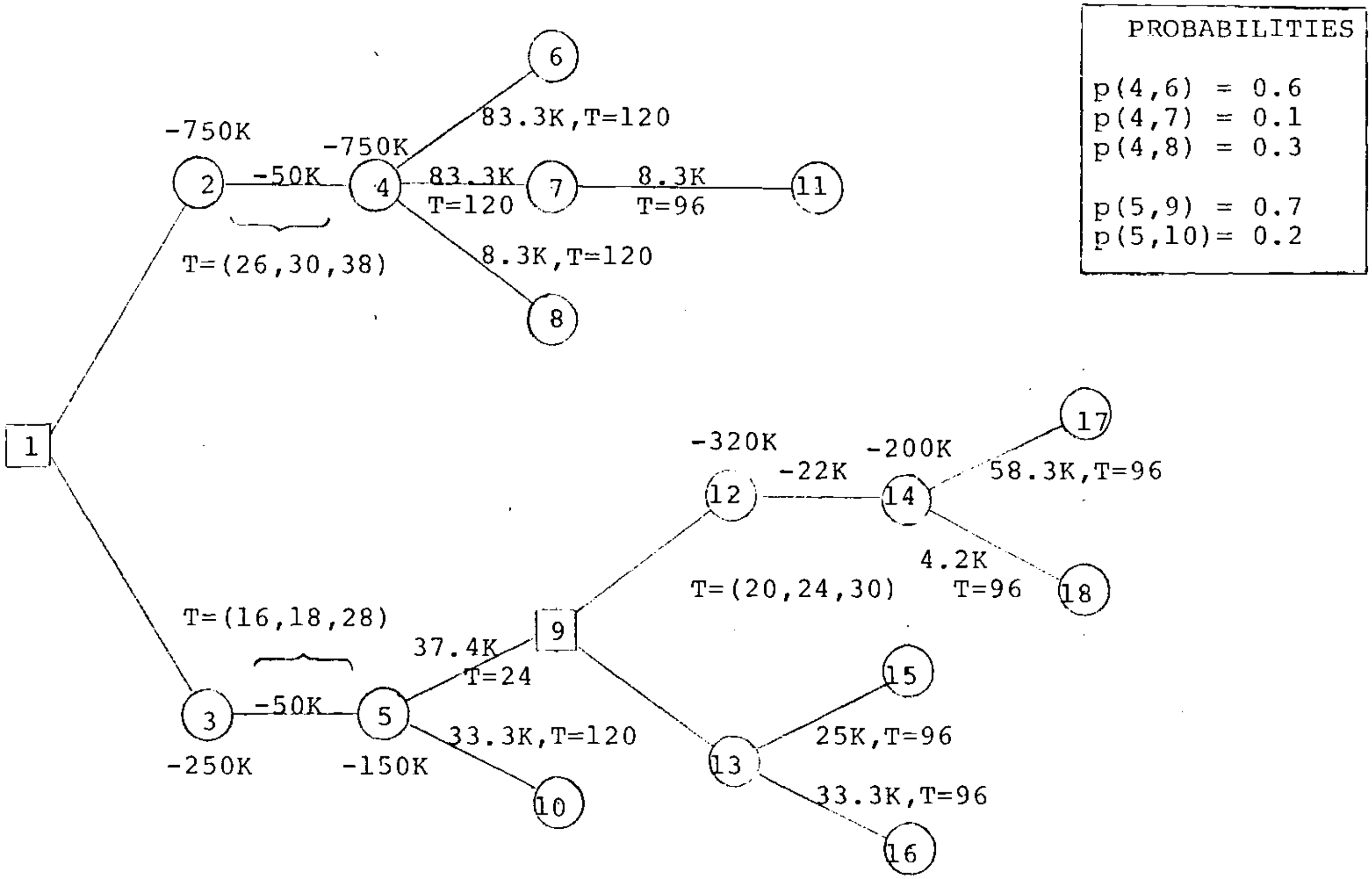


Figure 13. Decision Tree Diagram for Plant Construction Problem.

Table 14. Data for the Plant Construction Problem

	Probability	Revenue/Mo.	Time/Mo.
Alternative 1. Build Large Plant Initially			
High Demand	0.6	83.3K	120
High Demand Followed by Low Demand	0.1	83.3K 8.3K	24 96
Low Demand	0.3	8.3K	120
Alternative 2. Build Small Plant Initially			
High Demand	0.7	37.5K	24
Low Demand	0.3	33.3K	120
Alternative 2.1 Expand Plant			
High Demand	0.86	58.3K	96
Low Demand	0.14	4.2K	96
Alternative 2.2 No Plant Expansion			
High Demand	0.86	33.3K	96
Low Demand	0.14	25.0K	96

could be completed in 26 months and under adverse conditions it may take as long as 38 months. Similarly, if the smaller facility is constructed initially, an initial outlay of $250K is required, progress payments of $50K per month will have to be paid, and a termination cost of $150K will be incurred. It is estimated that this project will take between 16 months and 28 months for completion with a most-likely time to completion of 18 months. If the decision is made to construction of the smaller plant, node 9 represents the decision point where expansion of the plant is possible. This endeavor is estimated to require from 20 months to 30 months with an expected completion time of 24 months. An initial outlay of $320K and a termination charge of $200K will be required in addition to monthly progress payments of $22K for the duration of the project.

The worksheet used for input of this problem to the GTREE program is provided on the next page. An annual interest rate of 10% is assumed. Under the original problem formulation, without consideration of the construction time delay, the optimal decision was to construct the larger plant initially for a net present worth of $1,470K. The modified problem, allowing for variable construction time, yields an optimal decision to construct the smaller facility and expand the plant later if demand is high. The expected present worth of this option is $736K with a standard deviation of $93.5K. The option to construct the larger facility initially yields an expected present worth of $590K with a standard deviation of $89.7K.

2.2 Bridge Construction Example

This problem was extracted and modified into a stochastic decision tree problem from "Probability, Statistics, and Decision for Civil

WORKSHEET FOR DECISION TREE PROGRAM

ARC IDENTIFICATION		TERMINAL CASH FLOW			CONT. CASH FLOW			TIME DURATION				PROB- ABILITY
Source Node #	Sink Node #	999 LB	Mean UB	Var. MODE	999 LB	Mean UB	Var. MODE	DT	999 LB	Mean UB	Var. MODE	
1	2	999	-750	0								1.0
1	3	999	-250	0								1.0
2	4	999	-750	0	999	-50	0	A	26	38	30	1.0
3	5	999	-150	0	999	-50	0	A	16	28	18	1.0
4	6				999	83.3	0	A	999	120	0	0.6
4	7				999	83.3	0	A	999	24	0	0.1
4	8				999	8.3	0	A	999	120	0	0.3
5	9				999	37.5	0	A	999	24	0	0.7
5	10				999	33.3	0	A	999	120	0	0.3
7	11				999	8.3	0	A	999	96	0	1.0
9	12	999	-320	0								1.0
9	13											1.0
12	14	999	-200	0	999	-22	0	A	20	30	24	1.0
13	15				999	25	0	A	999	96	0	0.86
13	16				999	33.3	0	A	999	96	0	0.14
14	17				999	58.3	0	A	999	96	0	0.86
14	18				999	4.2	0	A	999	96	0	0.14

Definitions: LB = Lower bound
UB = Upper bound
MODE = Most likely value
999 = Specifies that mean and variance will be used.

DT = Distribution type:
C = constant
R = rectangular
E = exponential
G = gamma
N = normal
A = arbitrary

Engineers" by Benjamin and Cornell [3], pp. 547-553. The modified problem definition is as follows:

> A bridge is required to cross the John Day River in Oregon. The design specifications call for a structure that will withstand a river flow of 37,000 cfs. The plans called for the river to be layed in bedrock, but, due to excavation difficulties, the contractor would like to lay the foundation in compacted sand and gravel. This will yield cost savings based on an anticipated reduction in construction time by 3.3%. Furthermore, the expected variance in the construction time for the contractor's desired option is much less than that for the other option. To complete the project in bedrock is expected to take 3 years with a standard deviation of 1.18 years, while for the sand/gravel foundation option the expected time to completion is 2.9 years with a variance of 0.34 years squared. In the past, floods have caused flows in excess of 43,000 cfs. It is unlikely that a sand/gravel foundation would withstand the force of such a river flow. In approximately 6 years, however, the John Day River Dam should be completed placing the bridge over a calm lake. The decision problem is therefore whether or not to allow the contractor to found the pier in sand and gravel.

For certain flood levels, Table 16 indicates the likelihood of damage to the structure for each foundation option, and Table 15 indicates the probability of occurrence for each flood level. The initial start-up cost on the project is $640K with a variance of 4000K dollars squared, and the estimated termination charge is \$160K with a variance of 600K dollars squared. The monthly charge during the construction period is \$600K with a variance of 360K dollars squared. The decision tree structure for the problem is presented in Figure 14.

The evaluation of this decision tree indicates that both alternatives provide an equivalent expected present worth. For an interest rate of 12%, the sand/gravel option yields an expected present worth of -\$2,243K while the bedrock option yields an expected present worth of -\$2,248K. The standard deviation associated with each present worth is significantly different however. For the sand/gravel decision, the standard deviation is \$300K while for the bedrock decision it is \$532K. This implies that for a decision maker that is adverse to risk, the sand/gravel option would be preferable.

Table 15. Probability of Occurrence by Flood Level for the Bridge Construction Problem

Flood Level (cfs)	34,000	37,000	40,000	43,000
Probability of Occurrence	0.965	0.020	0.010	0.005

2.3 Computer Selection Example

The final example problem is a model of an actual decision problem undergoing consideration as of this writing. The situation is as follows:

> A large Southeastern corporation has exhausted the power of their computer resources and must make the decision to either upgrade their existing equipment or purchase a new

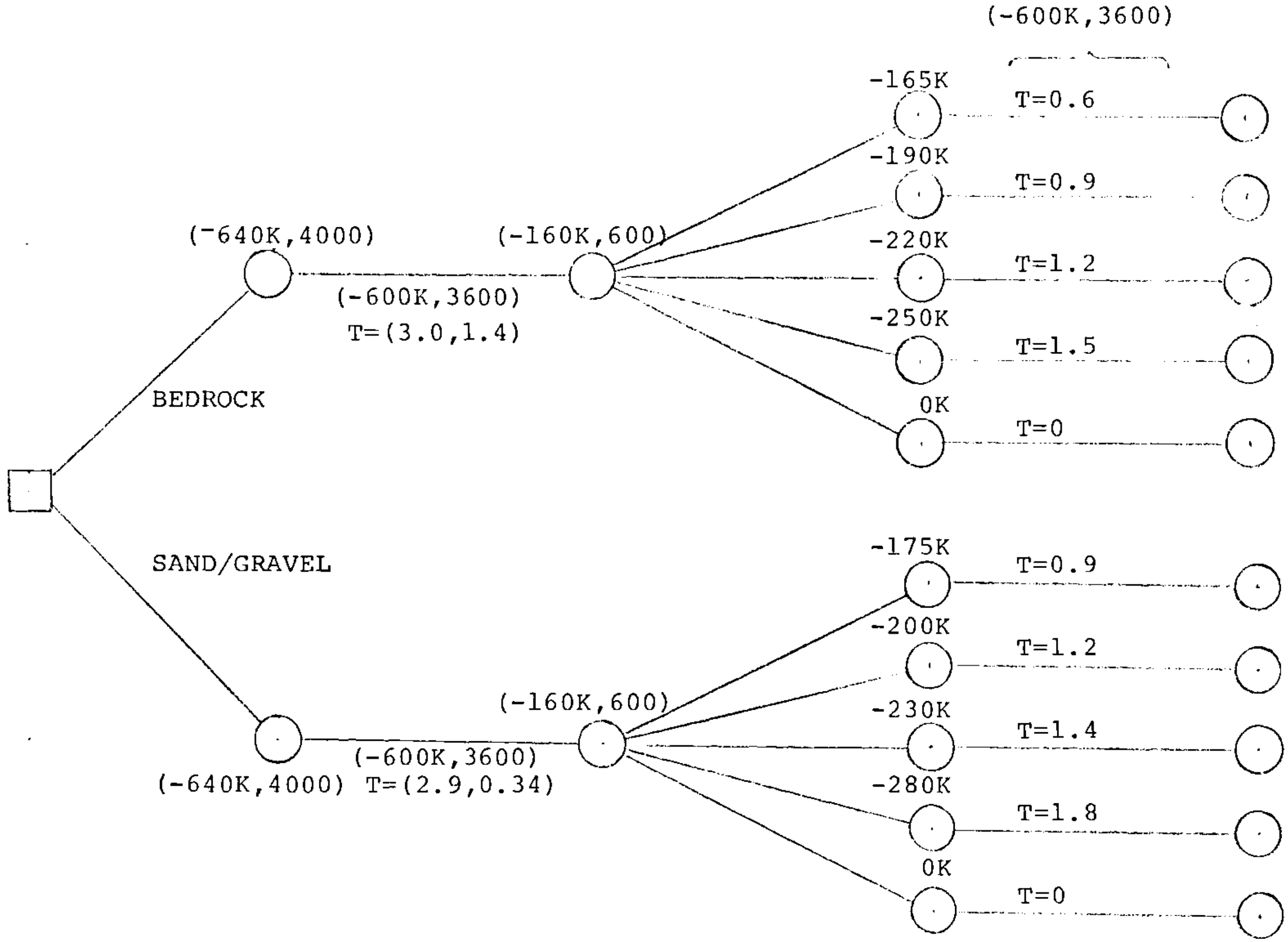


Figure 14. Decision Tree Diagram for Bridge Construction Problem.

Table 16. Probability and Extent of Damage by Maximum Flood Level for each Option in the Bridge Construction Problem

FOUNDATION IN BEDROCK

	Maximum Flood Level (cfs)			
	34,000	37,000	40,000	43,000
Bridge State				
Satisfactory	0.999	0.990	0.980	0.900
Not Satisfactory	0.001	0.010	0.020	0.100
Estimated Damage	$165,000	$190,000	$220,000	$250,000
Estimated time to Repair	0.6	0.9	1.1	1.4

FOUNDATION IN SAND/GRAVEL

	Maximum Flood Level (cfs)			
	34,000	37,000	40,000	43,000
Bridge State				
Satisfactory	0.990	0.900	0.500	0.200
Not Satisfactory	0.010	0.100	0.500	0.800
Estimated Damage	$175,000	$200,000	$230,000	$280,000
Estimated Time to Repair	0.9	1.2	1.4	1.8

machine which has been recently announced. The current system consists of a large 1973 vintage computer with six megabytes of memory and six block multiplexer channels. Initial sales quotations indicate that this existing hardware could be sold for \$2.0M. The corporation must decide whether to upgrade this owned equipment by adding two megabytes of memory, two additional channels, and installing an attached processor (AP) which would add approximately 60% to the processing capacity of the system, or to sell the existing equipment and acquire the new machine. The new machine provides 12 channels, 8 megabytes of memory, and an 80% capacity improvement over the current hardware as standard features. For each of these alternatives the corporation must decide whether to lease or purchase the additional equipment. In anticipation of this decision, the corporation has placed the necessary equipment for both options on order with the vendor. The channels and memory for the upgrade are scheduled to arrive at the same time as the new processor. The attached processor (AP) has a delivery position six months after that date.

The decision tree diagram for this problem is presented in Figure 15.

If the corporation chooses to upgrade the existing equipment, it has the option to either lease the two additional channels or purchase them outright. The memory upgrade must be purchased. If the channels are leased an initial outlay of \$258,941 will be required for supporting

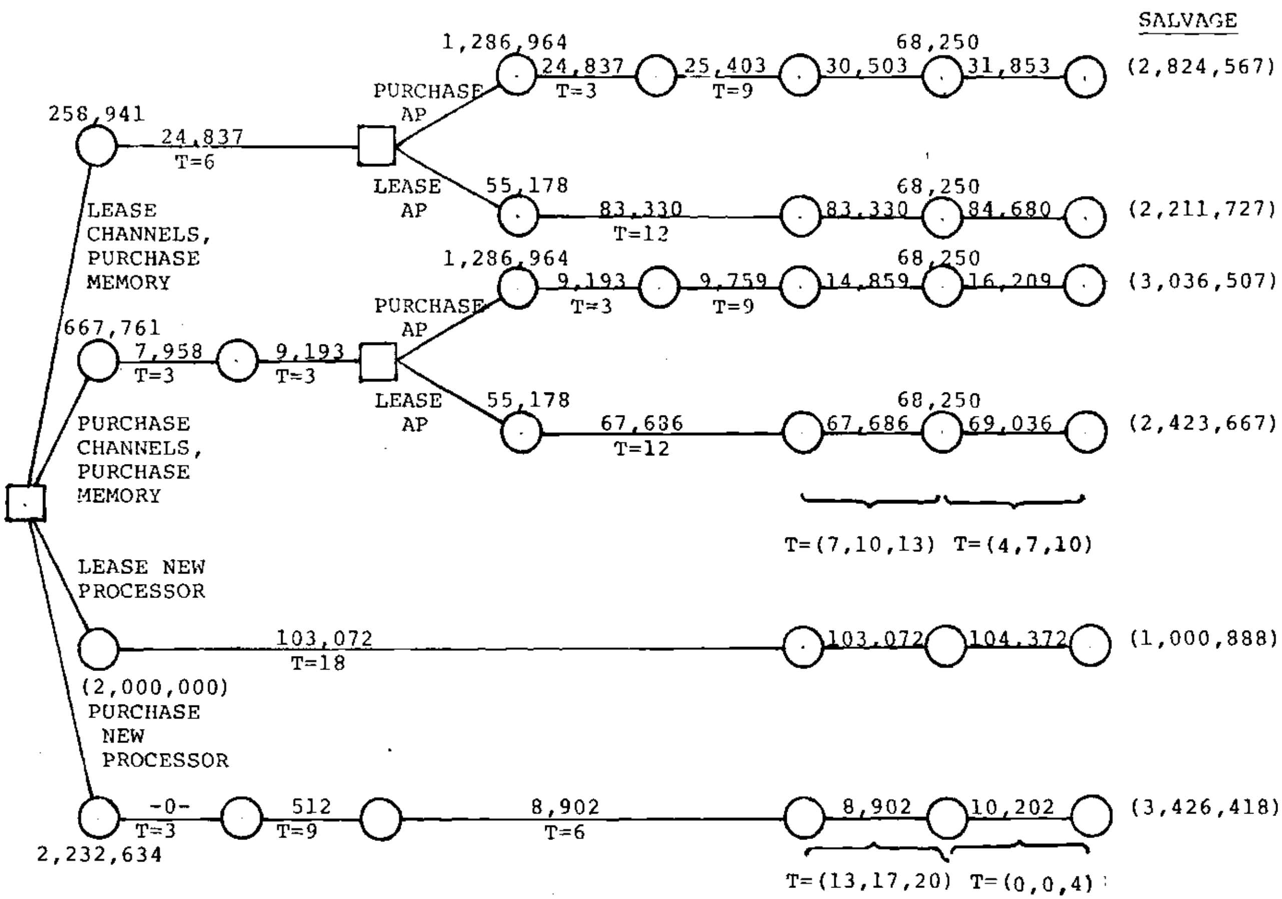


Figure 15. Decision Tree Diagram for the Computer Selection Problem.

equipment and a new monthly lease/maintenance cost of \$24,837 will be incurred. If the channels are purchased, an initial outlay of \$667,761 will be incurred. Under this latter option, the new monthly maintenance cost will be \$7,958 for three months while the channels are under warrantly and increase to \$9,193 thereafter.

After this six month period they must decide whether to lease or purchase the AP. The purchase price is \$1,286,964. The monthly charges will increase by \$566 after the three month warranty is exhausted on some of the support equipment and will increase again by \$5,100 when the 12 month warranty on the AP is exhausted. Should they decide to lease the AP, an initial outlay of \$55,178 will be required to purchase some support equipment for the owned computer and the monthly charge will increase by \$58,493.

If the decision is made to acquire the new processor, it may either by leased for \$103,072 per month or purchased for \$4,232,634. Under the purchase option there will be no monthly maintenance charge for the first three months while all of the equipment is under warranty. After three months, the maintenance charge will increase to \$512 and after 12 months it will increase to \$8,902. It is assumed that the existing equipment will be sold for \$2.0M if the new processor is acquired.

The estimated salvage values at the end of the 35 month planning horizon are shown in Figure 15. These figures include the estimated yield due to accruals on leased equipment.

Along with the announcement of the new processor, the vendor also announced the availability of an additional software/hardware feature called Systems Extension (SE) which, if installed, may enhance the capacity of a machine by 15%. The SE hardware is a standard feature of the new computer, but requires an outlay of $68,250 for installation on the older computer. The cost for the software is $1,300 monthly. A $50 per month maintenance charge is also applicable if the SE hardware is installed on the older computer. Therefore, if this feature is desired at a future date, it will cost $68,250 plus an additional $1,350 per month if the upgrade to the older computer is selected, or only $1,300 monthly if the new processor is selected.

The growth in processor requirements for the next few years is estimated to be between 20% and 25% per year. The expected growth is 22% per year. Based on this growth rate, the requirements relative to the base year (1978=100) are as follows for the next several years:

1978	1979	1980	1981	
100	120	144	173	20% growth rate
100	122	149	182	22% growth rate
100	125	156	195	25% growth rate

The AP complex will provide for a relative processor capacity of 160 without the SE and 184 with the SE. The new processor option will provide for the relative capacity of 180 without the SE and 207 with the SE. The anticipated growth pattern therefore implies that for each configuration option the SE will have to be installed according to the following time table:

	Time until SE Installed (Months)		
	Min	Exp	Max
AP Configuration	25	28	31
New Computer Configuration	31	35	38

Due to the planning horizon of 35 months, this implies that if the upgrade option is selected, the SE will be installed for an expected duration of 7 months with a minimum duration of 4 months and a maximum duration of 10 months. If the new computer option is selected the SE would be used from zero to 4 months with an expected duration of zero month.

The evaluation of this decision tree yields the results presented in Table 17. The optimal decision can be seen to be the purchase of the new computer. This provides for the greatest expected present worth and also a small variance.

Table 17. Results for the Computer Selection Problem

Decision Point 1	Decision Point 2	Expected Present Worth	Standard Deviation
Lease Channels, Purchase Memory	Purchase AP	(122,243)	25,002
Purchase Channels, Purchase Memory	Purchase AP	162,803	15,467
Lease New Computer	N/A	(515,264)	102,345
Purchase New Computer	N/A	446,348	19,183

2.4 Remarks Concerning Example Problems

The three example problems presented in this section exemplify the advantages to be gained from using random variables to describe the duration of activities in a decision tree. In the first example, the inclusion of variable construction times lead to the selection of a different decision than would have otherwise been made. In the second example, the inclusion of a variance factor for construction time provided the basis for a rational decision based on the reported variance of the expected present worth of each alternative. The third example demonstrated the flexibility provided by variable activity durations in modeling a problem where growth is uncertain.

Decision making is an art requiring significant judgement on the part of the decision maker. This fact may be profoundly demonstrated in the second example. The optimal choice was construction in compacted sand and gravel. This choice was selected by the authorities in Washington. Within a couple of years after the completion of the project, a devastating flood destroyed the bridge. Nevertheless, the proper decision has been made given the information available. The inclusion of additional variables into the model such as random timing with time-variance should provide additional insight to better enable decision makers to conclude the correct decisions.

CHAPTER VI

CONCLUSIONS

This thesis develops and demonstrates an extension to decision tree methodology for decision problems where activity durations are independent random variables, and where rewards or costs may occur uniformly throughout an activity's duration. The extended model is called a generalized decision tree. The derivation of the recursive relationships necessary to evaluate the expected present worth of a series of such activities, as well as the relationship to determine the variance of this series, are provided. These recursive relationships are summarized in Figure 9.

The solution procedure is based on a series of exact and approximate expressions for $E\{\beta^T\}$ presented by Young and Contreras [55] and a variance relationship provided by Rosenthal [43]. In Chapter IV, the accuracy of the approximate expressions is demonstrated. For practical ranges of parameters, it is first demonstrated that the approximations are accurate for individual activities; then for series of activities; and finally for use in a decision tree. It is demonstrated that, given liberal limitations on the skewness of the time distributions and the maximum allowable interest rate, the likelihood of an invalid path selection is at most 0.03 with a maximum expected error associated with this invalid selection of 0.3%. Therefore, the expected cost of a wrong decision is demonstrated to be less than 0.009%. Such an error is less

1. $$E\{S_k\} = \mu_k$$

$$E\{S_\eta\} = \mu_\eta + \omega_{\eta+1}\nu_{\eta+1} + E\{S_{\eta+1}\}\alpha_{\eta+1}$$

$$\text{for } \eta = k-1, k-2, \ldots, 1, 0$$

2. $$Q_k = \sigma_k^2$$

$$Q_\eta = \sigma_\eta^2 + (S_{\eta+1}^2 + Q_{\eta+1})\psi_{\eta+1} - S_{\eta+1}^2\alpha_{\eta+1}^2$$

$$+ (\zeta_{\eta+1}^2 + \omega_{\eta+1}^2)(\psi_{\eta+1} - \alpha_{\eta+1}^2)/r^2$$

$$+ \zeta_{\eta+1}^2\nu_{\eta+1}^2$$

$$\text{for } \eta = k-1, k-2, \ldots, 1, 0$$

Figure 16. Recursive Relationships for Expectation (1) and Variance (2).

than the significance of the model variables; an 'incorrect' path is for practical purposes an equally optimal path.

In Chapter V a computer program is developed to solve the generalized decision tree problem. The program provides extensive user prompting for ease of use. The program's menu selection format provides for the user the flexibility to access various modules for model entry and modification, data storage and retrieval, instruction, and execution. The program allows variables to be entered either explicitly as mean and variance or in the more intuitive approach using bounds and the mode. Three example problems are provided which depict the modeling usefulness of random timing for decision tree problems.

The conclusions, then, are

1. The generalized decision tree problem has an exact solution.
2. It has an accurate approximate solution requiring only the means and variances of activity durations rather than full distributions.
3. Solutions are obtainable at small computational effort using a computer program that has been tested and documented.
4. The generalized decision tree problem has modeling usefulness and reasonable data requirements.

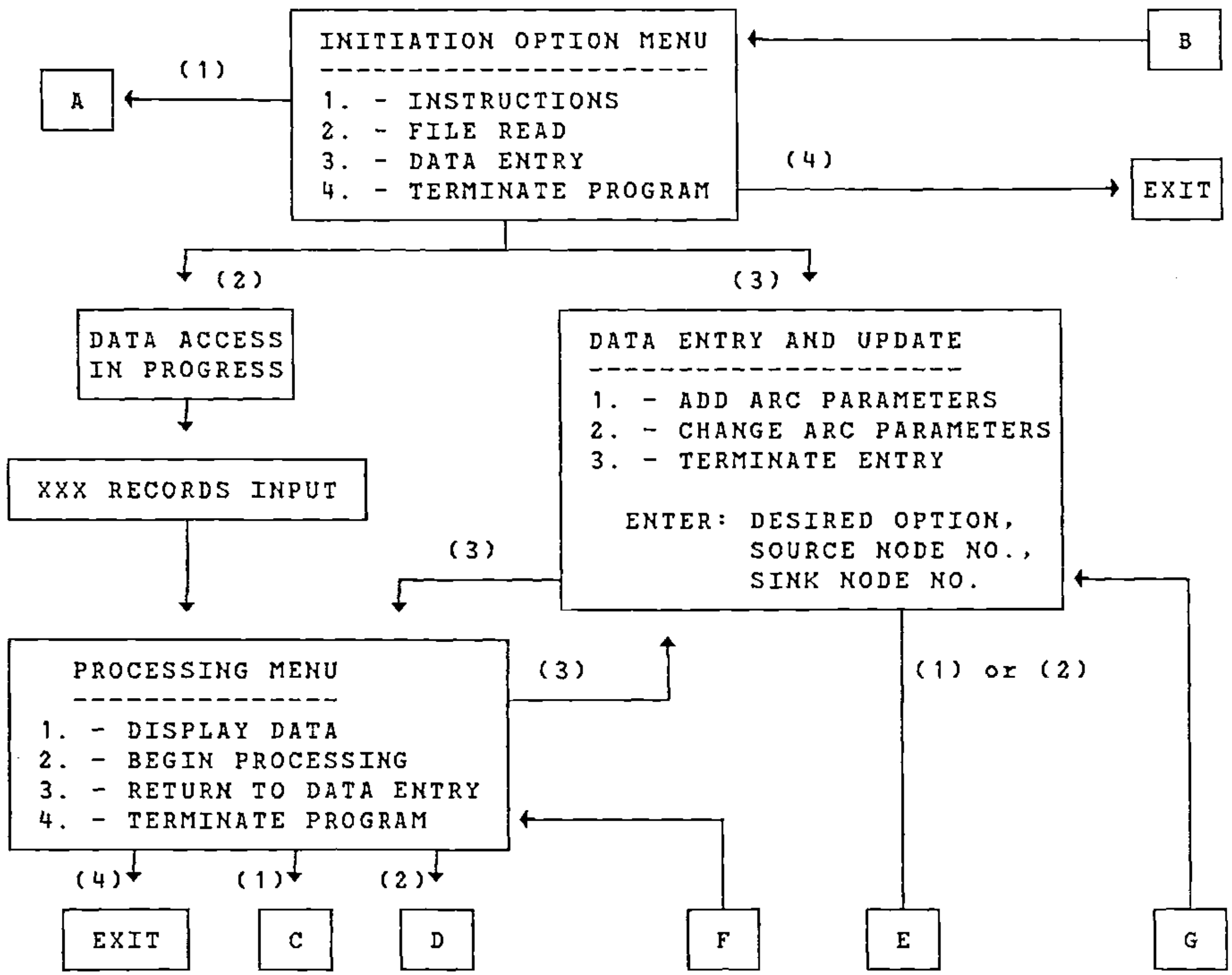
The most significant limitation of generalized decision trees is that activity durations are assumed to be independent random variables. In practice this is often a poor assumption [42]. Activity durations tend to be positively correlated, and to be correlated with costs. As Hillier has shown [25,26], analytical methods for independent contributions to an expected present worth can be extended to cover the case in which contributions are completely correlated; for incomplete correlation it has usually been necessary to resort to simulation [8,21]. However, it may be possible to apply analytical extensions to the generalized decision tree problem. Of particular interest would be the statistical theory of Ringer [42], which specifically treats (in a PERT context) interdependent activity durations. Investigation of the possibility of extending generalized decision trees to cover correlated activity durations is a recommended area for further research.

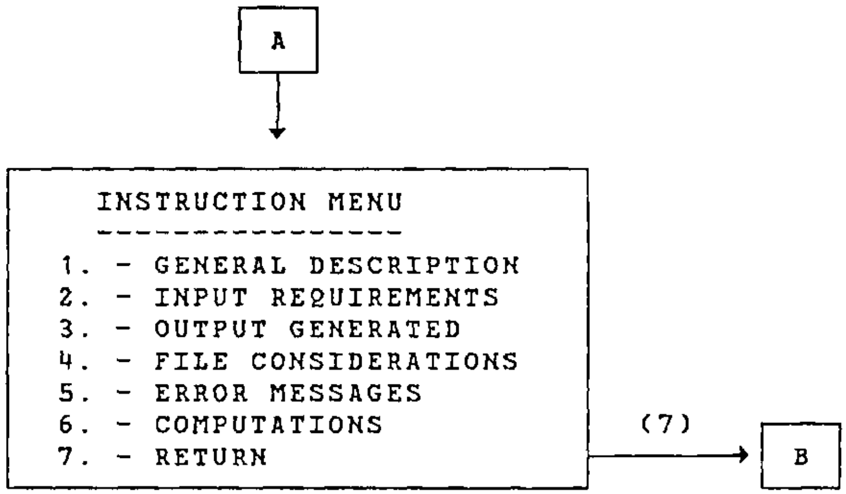
Extension of exact solutions to cover additional distributions of activity duration not covered in this thesis would present no difficulty, since $E\{\beta^T\}$ is known for any distribution of T having a known characteristic function, Laplace transform or moment generating function.

No difficulty would be encountered in extending the model to include exponentially increasing or decreasing costs throughout an activity, linear gradient costs, or arbitrary cost profiles. The model now allows a uniform cost throughout an activity; using the method of Young and Contreras [55], any cost profile could be converted to an equivalent cost at the start of the activity.

The conclusion regarding accuracy of the approximation for arbitrary activity duration distributions should carry over to the solution of semi-Markov decision processes. (A semi-Markov decision process is like a Markov decision process except that the time between transitions is a random variable.) The advantage would be in reducing the data requirements for such problems, since the time between transitions could be characterized by mean and variance rather than by the entire distribution. This should be investigated in further work.

APPENDIX


INITIATION OPTION MENU
1. - INSTRUCTIONS
2. - FILE READ
3. - DATA ENTRY
4. - TERMINATE PROGRAM
(1)
A
B
(4)
EXIT
(2)
(3)
DATA ACCESS IN PROGRESS
XXX RECORDS INPUT
DATA ENTRY AND UPDATE
1. - ADD ARC PARAMETERS
2. - CHANGE ARC PARAMETERS
3. - TERMINATE ENTRY
ENTER: DESIRED OPTION, SOURCE NODE NO., SINK NODE NO.
(3)
PROCESSING MENU
1. - DISPLAY DATA
2. - BEGIN PROCESSING
3. - RETURN TO DATA ENTRY
4. - TERMINATE PROGRAM
(3)
(1) or (2)
(4)
(1)
(2)
EXIT
C
D
F
E
G


A
INSTRUCTION MENU
1. - GENERAL DESCRIPTION
2. - INPUT REQUIREMENTS
3. - OUTPUT GENERATED
4. - FILE CONSIDERATIONS
5. - ERROR MESSAGES
6. - COMPUTATIONS
7. - RETURN
(7)
B

C

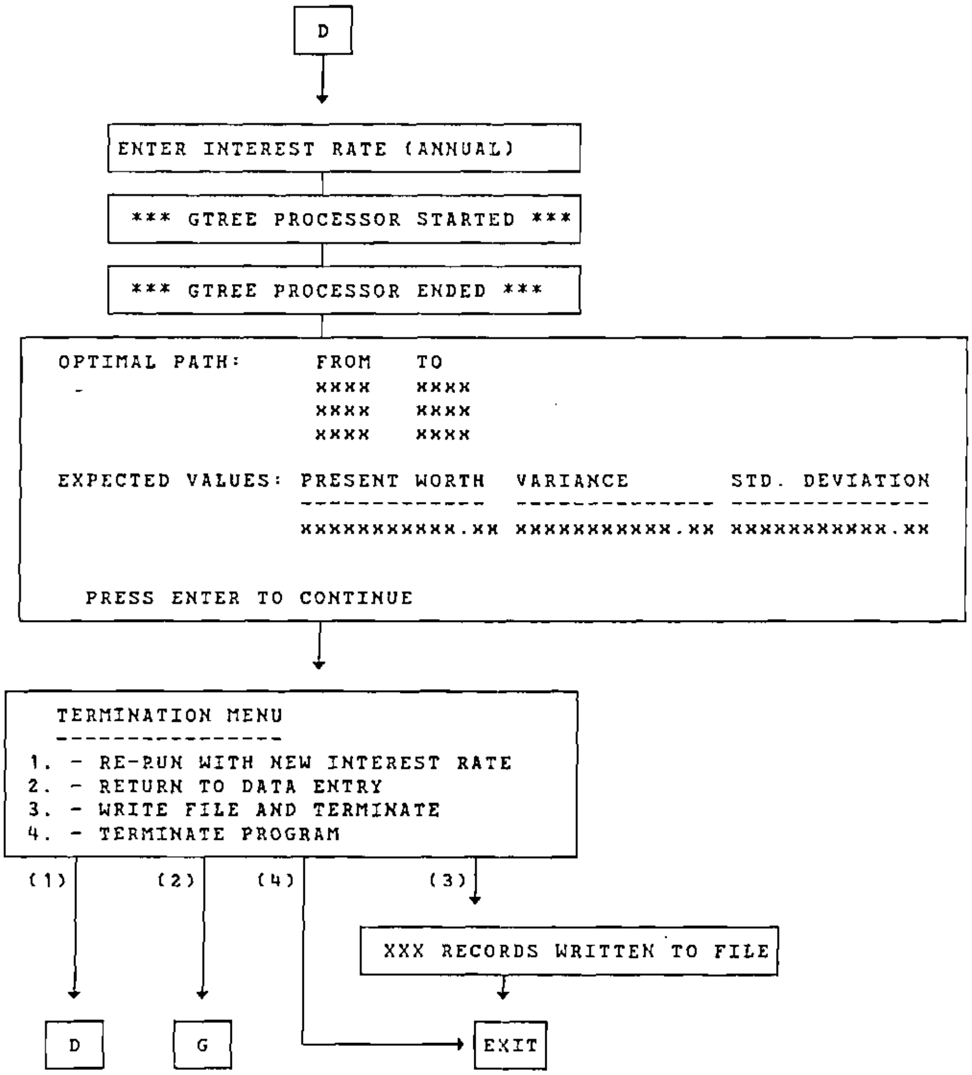
```
   SELECT DISPLAY OPTION
   ---------------------
1. - FROM NODE
2. - TO NODE
3. - DISPLAY ALL ARCS

  ENTER: OPTION,NODE(3,0) FOR ALL)
```

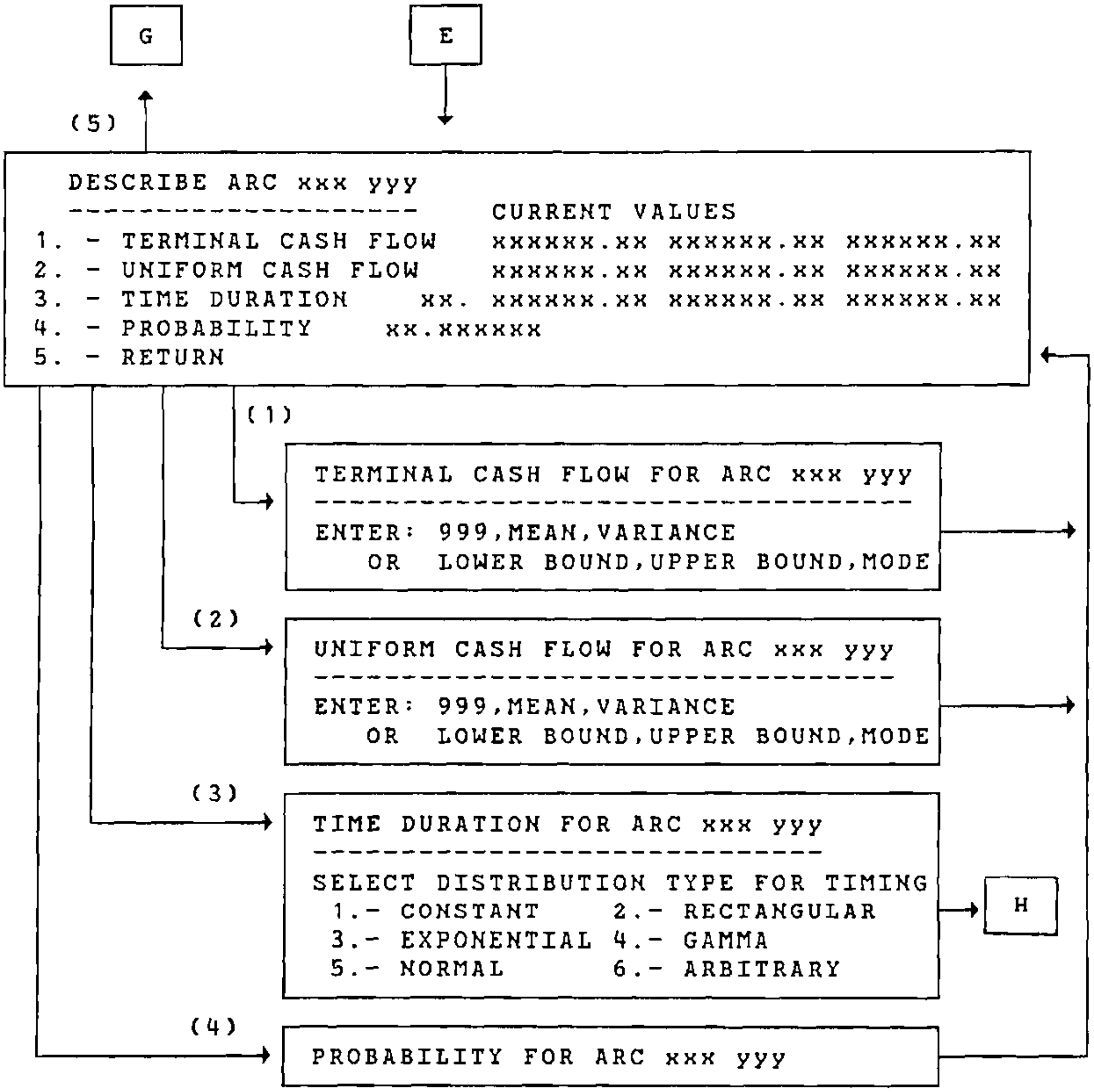
```
   DISPLAY OF ARCS NNNN XXXX
   -------------------------
      --TERM. CASH FLOW-- --CONT. CASH FLOW-- ---TIME DURATION ----
       999  MEAN  VAR.     999  MEAN  VAR.   DT  999  MEAN  VAR.
NNNN  LOWER UPPER MODE    LOWER UPPER MODE      LOWER UPPER MODE  PROB
----  ----- ----- -----   ----- ----- -----  -- ----- ----- ----  ----
XXXX  XXX.X XXX.X XXX.X   XXX.X XXX.X XXX.X  XX XXX.X XXX.X XXX.X X.XX
```

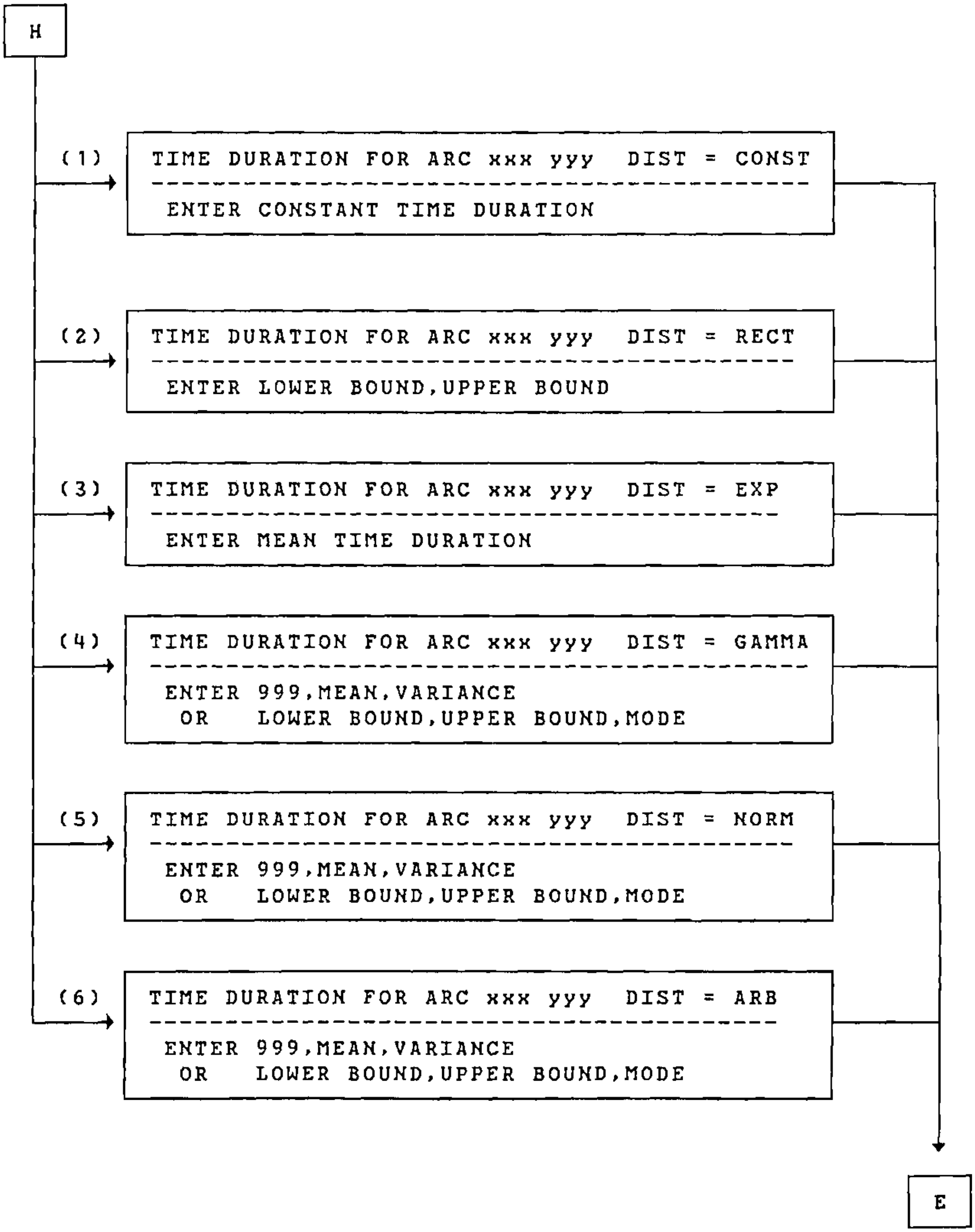
F

where NNNN equals either "from" or "to " depending
upon the option selected;

XXXX equals the node number.


D
ENTER INTEREST RATE (ANNUAL)
*** GTREE PROCESSOR STARTED ***
*** GTREE PROCESSOR ENDED ***
OPTIMAL PATH: FROM TO
XXXX XXXX
XXXX XXXX
XXXX XXXX
EXPECTED VALUES: PRESENT WORTH VARIANCE STD. DEVIATION
XXXXXXXXXXX.XX XXXXXXXXXXX.XX XXXXXXXXXXX.XX
PRESS ENTER TO CONTINUE
TERMINATION MENU
1. - RE-RUN WITH NEW INTEREST RATE
2. - RETURN TO DATA ENTRY
3. - WRITE FILE AND TERMINATE
4. - TERMINATE PROGRAM
(1)
(2)
(4)
(3)
XXX RECORDS WRITTEN TO FILE
D
G
EXIT


G
E
(5)
DESCRIBE ARC xxx yyy
CURRENT VALUES
1. - TERMINAL CASH FLOW xxxxxx.xx xxxxxx.xx xxxxxx.xx
2. - UNIFORM CASH FLOW xxxxxx.xx xxxxxx.xx xxxxxx.xx
3. - TIME DURATION xx. xxxxxx.xx xxxxxx.xx xxxxxx.xx
4. - PROBABILITY xx.xxxxxx
5. - RETURN
(1)
TERMINAL CASH FLOW FOR ARC xxx yyy
ENTER: 999,MEAN,VARIANCE
OR LOWER BOUND,UPPER BOUND,MODE
(2)
UNIFORM CASH FLOW FOR ARC xxx yyy
ENTER: 999,MEAN,VARIANCE
OR LOWER BOUND,UPPER BOUND,MODE
(3)
TIME DURATION FOR ARC xxx yyy
SELECT DISTRIBUTION TYPE FOR TIMING
1.- CONSTANT 2.- RECTANGULAR
3.- EXPONENTIAL 4.- GAMMA
5.- NORMAL 6.- ARBITRARY
H
(4)
PROBABILITY FOR ARC xxx yyy


H
(1)
TIME DURATION FOR ARC xxx yyy DIST = CONST
ENTER CONSTANT TIME DURATION
(2)
TIME DURATION FOR ARC xxx yyy DIST = RECT
ENTER LOWER BOUND,UPPER BOUND
(3)
TIME DURATION FOR ARC xxx yyy DIST = EXP
ENTER MEAN TIME DURATION
(4)
TIME DURATION FOR ARC xxx yyy DIST = GAMMA
ENTER 999,MEAN,VARIANCE
OR LOWER BOUND,UPPER BOUND,MODE
(5)
TIME DURATION FOR ARC xxx yyy DIST = NORM
ENTER 999,MEAN,VARIANCE
OR LOWER BOUND,UPPER BOUND,MODE
(6)
TIME DURATION FOR ARC xxx yyy DIST = ARB
ENTER 999,MEAN,VARIANCE
OR LOWER BOUND,UPPER BOUND,MODE
E

```
      PROGRAM MAIN(TAPE3,TAPE4,INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C   PROGRAM = GTREE  (GENERALIZED DECISION TREE PROGRAM)
C
C   WRITTEN BY : CARL H. WOHLERS
C
C   DATE: MARCH 17, 1978
C
C   IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
C   THE DEGREE MASTER OF SCIENCE IN OPERATIONS RESEARCH
C
C   THESIS TITLE:
C
C    DECISION TREES WITH INDEPENDENT STOCHASTIC ACTIVITY DURATIONS
C
C
C   ADVISORS:  DR. DONOVAN B. YOUNG, CHAIRMAN
C              DR. RONALD R. RARDIN
C              DR. GUNTER P. SHARP
C
C   DESCRIPTION: THIS PROGRAM IS WRITTEN IN STANDARD FORTRAN-IV
C                AND WILL SOLVE THE GENERALIZED DECISION TREE
C                PROBLEM WHERE THE DURATION OF ACTIVITIES AND
C                MAGNITUDE OF CASH FLOWS ARE INDEPENDENT
C                RANDOM VARIABLES.
C
C *******************************************************
C   DEFINE VARIABLES
C *************************************************
      DIMENSION S(50),Q(50)
      COMMON DTWORK(50,10),DTFILE(50,13)
      INTEGER PCNT,COUNT,PATH(50,2),SRCE,SINK,ARC
C *************************************************
C   DEFINE FUNCTIONS
C *************************************************
      PMEAN(A,B,C)=(A+4*C+B)/6.0
      PVAR(A,B)=(B-A)**2/36.0
      CONST(A,R)=EXP(-R*A)
      RECT(A,B,R)=(EXP(-R*A)-EXP(-R*B))/R*(B-A)
      EXPNT(A,R)=1/(1+A*R)
      GAMMA(A,B,R)=(1+(B*R/A))**(-A**2/B)
      NORM(A,B,R)=EXP(-A*R+.5*B*R**2)
      ARB(A,B,R)=EXP(-R*A)*(1+.5*B*R**2)
C **************************************************
C   INITIALIZATION
C **************************************************
      I=0
      DO 5 M=1,12
      DO 4 N=1,50
      DTFILE(N,M)=0.0
    4 CONTINUE
    5 CONTINUE
      DO 10 M=1,50
      DTFILE(M,13)=1.0
      S(M)=0.0
      Q(M)=0.0
   10 CONTINUE
C **************************************************
C   INITIAL OPTION MENU
C **************************************************
```

```
   14 WRITE(6,15)
   15 FORMAT(/2X,'INITIAL OPTION MENU',/2X,
     119('-'),//1X,
     2'1. - INSTRUCTIONS',/1X,
     3'2. - FILE READ',/1X,
     4'3. - DATA ENTRY',/1X,
     5'4. - TERMINATE PROGRAM')
      READ(5,16) J
   16 FORMAT(I1)
      IF (J.GT.4) GOTO 14
      GOTO (25,220,128,620),J
C ***********************************************************
C    INSTRUCTION MODULE
C ***********************************************************
   25 WRITE(6,26)
   26 FORMAT(//3X,'INSTRUCTION MENU',/3X,16('-'),//1X,
     1'1. - GENERAL DESCRIPTION',/1X,'2. - INPUT REQUIREMENTS',/1X,
     2'3. - OUTPUT GENERATED',/1X,'4. - FILE CONSIDERATIONS',/1X,
     3'5. - ERROR MESSAGES',/1X,'ANALYTICAL METHODOLOGY',/1X,
     4'7. - RETURN')
      READ(5,*) J
      IF (J.GT.7) GOTO 25
      GOTO (30,40,50,60,70,80,14),J
   30 WRITE(6,31)
   31 FORMAT(/1X,'GENERAL DESCRIPTION',
     1//1X,'THIS PROGRAM WILL SOLVE THE GENERALIZED DECISION TREE ',/1X,
     2'PROBLEM WHERE ACTIVITY DURATIONS ARE REPRESENTED BY RANDOM '/1X,
     3'VARIABLES. THESE VARIABLES MAY BE DESCRIBED BY EXPLICITLY'/1X,
     4'SPECIFYING THE MEAN AND VARIANCE OR BY SPECIFYING THE UPPER'/1X,
     5'AND LOWER BOUNDS WITH THE MODE.'//1X,'TWO TYPES OF CASH FLOWS',
     6' MAY BE DEFINED '/1X,'1) CASH FLOWS OCCURRING AT THE END',
     7/1X,'OF THE ACTIVITY, 2) CONTINUOUS UNIFORM CASH FLOWS THAT '/1X,
     8'ARE APPLICABLE FOR THE DURATION OF THE ACTIVITY. THESE VARIABLES',
     9' ARE'/1X,'DEFINED SIMILARLY TO THE TIME VARIABLES BY SPECIFYING',
     A' MEANS'/1X,'AND VARIANCES OR BOUNDS AND THE MODE.'//1X)
      WRITE(6,38)
   38 FORMAT(1X,'THE INDEPENDENCE OF ALL VARIABLES IS ASSUMED')
      GOTO 25
   40 WRITE(6,41)
   41 FORMAT(/1X,'INPUT REQUIREMENTS',
     1//1X,'THREE TYPES OF RANDOM VARIABLES ARE ')
      GOTO 25
   50 WRITE(6,51)
   51 FORMAT(/1X,'OUTPUT GENERATED')
      GOTO 25
   60 WRITE(6,61)
   61 FORMAT(/1X,'FILE CONSIDERATIONS')
      GOTO 25
   70 WRITE(6,71)
   71 FORMAT(/1X,'ERROR MESSAGES')
      GOTO 25
   80 WRITE(6,81)
   81 FORMAT(/1X,'ANALYTICAL METHODOLOGY')
      GOTO 25
C ***********************************************************
C    FILE READ MODULE
C ***********************************************************
  220 WRITE(6,225)
  225 FORMAT(/1X,'FILE ACCESS IN PROGRESS')
  227 I=I+1
```

```
      READ(3,*) (DTFILE(I,L),L=1,13)
      IF (EOF(3)) 228,227
  228 I=I-1
      WRITE(6,229) I
  229 FORMAT(1X,I3,' RECORDS INPUT FROM FILE')
      GOTO 250
C ************************************************
C   DATA ENTRY AND UPDATE
C ************************************************
  128 I4=I+1
      WRITE(6,130) I4
  130 FORMAT(//3X,'DATA ENTRY AND UPDATE',/3X,21('-'),//1X,
     1'1. - ADD ARC PARAMETERS (ARC= ',I2,')',/1X,
     2'2. - CHANGE ARE PARAMETERS',/1X,
     3'3. - TERMINATE ENTRY AND UPDATE',//3X,
     4'ENTER - DESIRED OPTION,SOURCE NODE NUMBER,SINK NODE NUMBER')
      READ(5,*) I3,I1,I2
      IF (I3.GT.3) GOTO 128
      GOTO (140,180,250),I3
C ************************************************
C   DATA DESCRIPTION MODULE
C ************************************************
  140 I=I+1
      DTFILE(I,1)=I1
      DTFILE(I,2)=I2
      IF (I.EQ.1) GOTO 142
      IF (DTFILE(I-1,1).LE.DTFILE(I,1)) GOTO 144
      WRITE(6,141) DTFILE(I,1),DTFILE(I-1,1)
  141 FORMAT(/3X,'GTE001 CURRENT COURSE NODE (',F3.0,') MUST BE',
     1' GREATER THAN OR EQUAL TO PREVIOUS SOURCE NODE (',F3.0,').',
     2' RE-ENTER.')
      I=I-1
      GOTO 128
  142 IF (DTFILE(I,1).EQ.1.0) GOTO 144
      WRITE(6,143)
  143 FORMAT(//2X,'GTE002 INITIAL SOURCE NODE MUST EQUAL 1')
      I=I-1
      GOTO 128
  144 IF (I.EQ.1) GOTO 150
      GOTO 147
      TESTV=DTFILE(I,1)-DTFILE(I-1,1)
      IF (TESTV.EQ.1.0.OR.TESTV.EQ.0.0) GOTO 147
      WRITE(6,145)
  145 FORMAT(//2X,'GTE003 SOURCE NODES MUST BE NUMBERED SEQUENTIALLY')
      I=I-1
      GOTO 128
  147 TESTV=DTFILE(I,2)-DTFILE(I-1,2)
      IF (TESTV.EQ.1.0) GOTO 150
      WRITE(6,148)
  148 FORMAT(//2X,'GTE004 SINK VALUES MUST BE NUMBERED SEQUENTIALLY')
      I=I-1
      GOTO 128
  150 WRITE(6,151) (DTFILE(I,L),L=1,13)
  151 FORMAT(//3X,'DESCRIBE ARC ',F3.0,1X,F3.0,':',/3X,20('-'),5X,
     1'CURRENT VALUES',/1X,'1. - TERMINAL CASH FLOW  ',
     23(F9.3,1X),/1X,'2. - UNIFORM CASH FLOW   ',3(F9.3,1X),/1X,
     3'3. - TIME DURATION    ',F2.0,1X,3(F9.3,1X),/1X,
     4'4. - PROBABILITY     ',F10.8,/1X,'5. - RETURN',//2X,
     5'ENTER DESIRED OPTION')
      READ(5,*) I3
```

```
      IF (I3.GT.5) GOTO 150
      GOTO (154,158,164,168,128),I3
  154 WRITE(6,155) DTFILE(I,1),DTFILE(I,2)
  155 FORMAT(//3X,'TERMINAL CASH FLOW FOR ARC ',F3.0,1X,F3.0,/3X
     134('-'),/1X,'ENTER: 999,MEAN,VARIANCE',/4X,
     2'OR  LOWER BOUND,UPPER BOUND,MODE')
      READ(5,*) DTFILE(I,3),DTFILE(I,4),DTFILE(I,5)
      GOTO 150
  158 WRITE(6,159) DTFILE(I,1),DTFILE(I,2)
  159 FORMAT(//3X,'CONTINUOUS UNIFORM CASH FLOW FOR ARC ',
     1F3.0,1X,F3.0,/3X,44('-'),//1X,
     2'ENTER: 999,MEAN,VARIANCE',/4X,
     3'OR  LOWER BOUND,UPPER BOUND,MODE')
      READ(5,*) (DTFILE(I,L),L=6,8)
      GOTO 150
  164 WRITE(6,165) DTFILE(I,1),DTFILE(I,2)
  165 FORMAT(//3X,'TIME DURATION FOR ARC ',F3.0,1X,F3.0,/3X,
     129('-'),//1X,'SELECT DISTRIBUTION TYPE FOR TIMING:',/3X,
     2'1. - CONSTANT',15X,'2. - RECTANGULAR',/3X,
     3'3. - NEGATIVE-EXPONENTIAL',3X,'4. - GAMMA',/3X,
     4'5. - NORMAL',17X,'6. - ARBITRARY')
      READ(5,*) I5
      IF (I5.GT.6) GOTO 164
      DTFILE(I,9)=I5
      CALL TIME(I5,I)
      GOTO 150
  168 WRITE(6,169) DTFILE(I,1),DTFILE(I,2)
  169 FORMAT(//3X,'ENTER PROBABILITY OF OCCURRENCE FOR ARC ',
     1F3.0,1X,F3.0,/3X,47('-'))
      READ(5,*) DTFILE(I,13)
      GOTO 150
C ********************************************************
C   DATA DESCRIPTION MODULE - UPDATE AN ARC
C ********************************************************
  180 IC=I2-1
      WRITE(6,181) (DTFILE(IC,L),L=1,13)
  181 FORMAT(//3X,'UPDATE ARC ',F3.0,1X,F3.0,' :',/3X,18('-'),6X,
     1'CURRENT VALUES',/1X,'1. - TERMINAL CASH FLOW    ',
     23(F9.3,1X),/1X,'2. - UNIFORM CASH FLOW     ',3(F9.3,1X),/1X,
     3'3. - TIME DURATION       ',F2.0,1X,3(F9.3,1X),/1X,
     4'4. - PROBABILITY      ',F11.9,/1X,'5. - RETURN',///2X,
     5'ENTER DESIRED OPTION')
      READ(5,*) J
      IF (J.GT.5) GOTO 184
      GOTO (186,190,194,198,128),J
  184 WRITE(6,185) J
  185 FORMAT(//2X,'GTE005 RESPONSE (',I1,') INVALID. RE-ENTER.')
      GOTO 180
  186 WRITE(6,187) DTFILE(IC,1),DTFILE(IC,2)
  187 FORMAT(///3X,'TERMINAL CASH FLOW FOR ARC ',F3.0,1X,F3.0,/3X,
     134('-'))
      WRITE(6,188) (DTFILE(IC,L),L=3,5)
  188 FORMAT(//1X,'CURRENT = ',3(F9.3,1X),//1X,
     1'ENTER NEW: 999,MEAN,VARIANCE  OR  L.B.,U.B.,MODE')
      READ(5,*) (DTFILE(IC,L),L=3,5)
      GOTO 180
  190 WRITE(6,191) (DTFILE(IC,L),L=1,2)
  191 FORMAT(//3X,'CONTINUOUS UNIFORM CASH FLOW FOR ARC ',F3.0,1X,F3.0,
     1/3X,44('-'))
      WRITE(6,192) (DTFILE(IC,L),L=6,8)
```

```
  192 FORMAT(//1X,'CURRENT = ',3(F9.3,1X),//1X,
     1'ENTER NEW: 999,MEAN,VARIANCE  OR  L.B.,U.B.,MODE')
      READ(5,*) (DTFILE(IC,L),L=6,8)
      GOTO 180
  194 WRITE(6,165) (DTFILE(IC,L),L=1,2)
      READ(5,*) I5
      DTFILE(IC,9)=I5
      CALL TIME(I5,IC)
      GOTO 128
  198 WRITE(6,199) (DTFILE(IC,L),L=1,2)
  199 FORMAT(//3X,'PROBABILITY FOR ARC ',F3.0,1X,F3.0,/3X,
     127('-'))
      WRITE(6,200) DTFILE(IC,13)
  200 FORMAT(/1X,'CURRENT VALUE = ',F4.2,//1X,'ENTER NEW VALUE')
      READ(5,*) DTFILE(IC,13)
      GOTO 180
C ********************************************************
C   PROCESSING MENU
C ********************************************************
  250 WRITE(6,255)
  255 FORMAT(//3X,'ENTER: 1. - TO DISPLAY DATA',/10X,
     1'2. - TO PROCEED WITH PROCESSING',/10X,
     2'3. - RETURN TO DATA ENTRY',/10X,'4. - TO TERMINATE PROGRAM')
  258 READ(5,260) J
  260 FORMAT(I1)
      IF (J.GT.4) GOTO 261
      GOTO (270,300,128,620),J
  261 WRITE(6,262) J
  262 FORMAT(//2X,'GTE005 RESPONSE (',I1,') INVALID. RE-ENTER.')
      GOTO 258
  270 CALL EDTDIS(J,1)
      GOTO 250
C ****************************************************
C   PRE-PROCESSOR
C ****************************************************
  300 WRITE(6,305)
  305 FORMAT(/1X,'ENTER INTEREST RATE FOR THIS RUN (ANNUAL)')
      READ(5,*) RATE
      RATE=ALOG(1+RATE)
      WRITE(6,310)
  310 FORMAT(//3X,'*** GTREE PRE-PROCESSOR STARTED ***')
      DO 390 KK=1,I
C
C   NODE TRANSLATION
C
      DTWORK(KK,1)=DTFILE(KK,1)
      DTWORK(KK,2)=DTFILE(KK,2)
C
C   TERMINAL CASH FLOW TRANSLATION
C
      IF (DTFILE(KK,3).NE.999.0) GOTO 315
      DTWORK(KK,3)=DTFILE(KK,4)
      DTWORK(KK,4)=DTFILE(KK,5)
      GOTO 320
  315 DTWORK(KK,3)=PMEAN(DTFILE(KK,3),DTFILE(KK,4),DTFILE(KK,5))
      DTWORK(KK,4)=PVAR(DTFILE(KK,3),DTFILE(KK,4))
C
C   CONTINUOUS CASH FLOW TRANSLATION
C
  320 IF (DTFILE(KK,6).NE.999.0) GOTO 325
```

```
      DTWORK(KK,5)=DTFILE(KK,7)
      DTWORK(KK,6)=DTFILE(KK,8)
      GOTO 330
  325 DTWORK(KK,5)=PMEAN(DTFILE(KK,6),DTFILE(KK,7),DTFILE(KK,8))
      DTWORK(KK,6)=PVAR(DTFILE(KK,6),DTFILE(KK,7))
C
C    TIME TRANSLATION
C
  330 R=RATE
      R2=2.*RATE
      IF (DTFILE(KK,10).EQ.999.0) GOTO 333
C
C    UPPER,LOWER,MODE TIME TRANSLATION
C
      VAL1=PMEAN(DTFILE(KK,10),DTFILE(KK,11),DTFILE(KK,12))
      VAL2=PVAR(DTFILE(KK,10),DTFILE(KK,11))
C
C    TIME TRANSLATION BASED ON DISTRIBUTION TYPE
C
  333 I3=INT(DTFILE(KK,9))
      IF (I3.EQ.0) GOTO 335
      GOTO (335,340,345,350,355,360),I3
  335 DTWORK(KK,7)=CONST(DTFILE(KK,10),R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=CONST(DTFILE(KK,10),R2)
      GOTO 370
  340 DTWORK(KK,7)=RECT(DTFILE(KK,10),DTFILE(KK,11),R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=RECT(DTFILE(KK,10),DTFILE(KK,11),R2)
      GOTO 370
  345 DTWORK(KK,7)=EXPNT(DTFILE(KK,10),R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=EXPNT(DTFILE(KK,10),R2)
      GOTO 370
  350 IF (DTFILE(KK,10).NE.999.0) GOTO 352
      DTWORK(KK,7)=GAMMA(DTFILE(KK,11),DTFILE(KK,12),R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=GAMMA(DTFILE(KK,11),DTFILE(KK,12),R2)
      GOTO 370
  352 DTWORK(KK,7)=GAMMA(VAL1,VAL2,R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=GAMMA(VAL1,VAL2,R2)
      GOTO 370
  355 IF (DTFILE(KK,10).NE.999.0) GOTO 357
      DTWORK(KK,7)=NORM(DTFILE(KK,11),DTFILE(KK,12),R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=NORM(DTFILE(KK,11),DTFILE(KK,12),R2)
      GOTO 370
  357 DTWORK(KK,7)=NORM(VAL1,VAL2,R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=NORM(VAL1,VAL2,R2)
      GOTO 370
  360 IF (DTFILE(KK,10).NE.999.0) GOTO 362
      DTWORK(KK,7)=ARB(DTFILE(KK,11),DTFILE(KK,12),R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=ARB(DTFILE(KK,11),DTFILE(KK,12),R2)
      GOTO 370
  362 DTWORK(KK,7)=ARB(VAL1,VAL2,R)
      DTWORK(KK,8)=(1.-DTWORK(KK,7))/R
      DTWORK(KK,9)=ARB(VAL1,VAL2,R2)
```

```
  370 DTWORK(KK,10)=DTFILE(KK,13)
  390 CONTINUE
C ****************************************************************
C   PROCESSOR MODULE
C ****************************************************************
      WRITE(6,395)
  395 FORMAT(//3X,'*** GTREE PRE-PROCESSOR ENDED ***',//3X,
     1'*** GTREE PROCESSOR STARTED ***')
C
C   STORE ARC COUNTER
C
  500 COUNT=I
      PCNT=1
C
C   DETERMINE: SOURCE NODE NUMBER ==> SRCE
C              SINK NODE NUMBER   ==> SINK
C              ARC NUMBER         ==> ARC
C
  502 SRCE=INT(DTWORK(I,1))
      SINK=INT(DTWORK(I,2))
      ARC = SINK -1
C
C   BRANCH TO 550 ON CHANCE ARC
C
      IF (DTWORK(ARC,10).LT.1.0) GOTO 550
C
C   DETERMINE SINK NODE VALUE
C
  505 S(SINK)=DTWORK(ARC,3) + S(SINK)
C
C   COMPUTE SOURCE NODE VALUE GIVEN THIS ARC SELECTED
C
      VALUE=DTWORK(ARC,5)*DTWORK(ARC,8)+S(SINK)*DTWORK(ARC,7)
C
C   IF SOURCE NODE VALUE =0, SELECT NEW VALUE
C
      IF (S(SRCE).EQ.0.0) GOTO 525
C
C   IF VALUE GREATER THAN PREVIOUS SOURCE NODE VALUE. SELECT IT
C
      IF (VALUE.LT.S(SRCE)) GOTO 530
C
C   SELECT THIS ARC
C
  525 S(SRCE)=VALUE
      PATH(PCNT,1)=SRCE
      PATH(PCNT,2)=SINK
C
C   BRANCH IF LAST ARC
C
  530 IF (I.EQ.1) GOTO 535
      I=I-1
      K=I+1
      IF (DTWORK(I,1).EQ.DTWORK(K,1)) GOTO 502
  535 SRCE=PATH(PCNT,1)
      SINK=PATH(PCNT,2)
      ARC=SINK-1
      Q(SINK)=DTWORK(ARC,4) + Q(SINK)
      VAL1 = (S(SINK)**2+Q(SINK))*DTWORK(ARC,9)
      VAL2 = (S(SINK)*DTWORK(ARC,7))**2
```

```
      VAL3 = DTWORK(ARC,6)+DTWORK(ARC,5)**2
      VAL4 = DTWORK(ARC,9)-DTWORK(ARC,7)**2
      VAL5 = DTWORK(ARC,6)*DTWORK(ARC,8)**2
      Q(SRCE) = VAL1-VAL2+(VAL3*VAL4/RATE**2)+VAL5
      IF (I.EQ.1) GOTO 600
      PCNT = PCNT + 1
      GOTO 502
  550 S(SINK)=DTWORK(ARC,3)+S(SINK)
      VALUE = DTWORK(ARC,5)*DTWORK(ARC,8)+S(SINK)*DTWORK(ARC,7)
      S(SRCE)=S(SRCE)+VALUE*DTWORK(ARC,10)
      Q(SINK)=Q(SINK)+DTWORK(ARC,4)
      VAL1=(S(SINK)**2+Q(SINK))*DTWORK(ARC,9)
      VAL2=(S(SINK)*DTWORK(ARC,7))**2
      VAL3=DTWORK(ARC,6)+DTWORK(ARC,5)**2
      VAL4=DTWORK(ARC,9)-DTWORK(ARC,7)**2
      VAL5=DTWORK(ARC,6)*DTWORK(ARC,8)**2
      QADD=VAL1-VAL2+(VAL3*VAL4/RATE**2)+VAL5
      QADD=QADD*DTWORK(ARC,10)**2
      Q(SRCE)=Q(SRCE)+QADD
      IF (I.EQ.1) GOTO 600
      I=I-1
      GOTO 502
C *********************************************************
C   DISPLAY MODULE
C *********************************************************
  600 I=0
      WRITE(6,601)
  601 FORMAT(//2X,'OPTIMAL DECISIONS:      FROM   TO')
  605 I=I+1
      IF (PATH(I,1).EQ.0) GOTO 625
      WRITE(6,615) PATH(I,1),PATH(I,2)
  615 FORMAT(24X,I3,3X,I3)
      GOTO 605
  625 WRITE(6,630)
  630 FORMAT(//2X,'EXPECTED VALUES:     MEAN      VARIANCE     STD.DEV.',
     1/22X,4('-'),5X,7('-'),5X,7('-'))
      VALUE=SQRT(Q(1))
      WRITE(6,635) S(1),Q(1),VALUE
  635 FORMAT(/17X,3(F10.2,2X))
      PAUSE 'PRESS ENTER TO CONTINUE'
C *********************************************************
C   TERMINATION MENU
C *********************************************************
  699 WRITE(6,700)
  700 FORMAT(//2X,'SELECT OPTION : ',/2X,14('-'),//1X,
     1'1. - RERUN WITH NEW INTEREST RATE',/1X,
     2'2. - RETURN TO DATA ENTRY/UPDATE',/1X,
     3'3. - SAVE FILE AND TERMINATE PROGRAM',/1X,
     4'4. - TERMINATE PROGRAM')
      READ(5,*) I5
      IF (I5.GT.4) GOTO 699
      GOTO (710,710,720,620),I5
C *********************************************************
C   INITIALIZATION FOR RE-RUN
C *********************************************************
  710 I=COUNT
      DO 715 II=1,50
      PATH(II,1)=0
      PATH(II,2)=0
      S(II)=0.0
```

```
      Q(II)=0.0
  715 CONTINUE
      GOTO (300,250),I5
  720 WRITE(6,725)
  725 FORMAT(//2X,'*** FILE WRITE IN PROGRESS ***')
      REWIND 4
      DO 730 J=1,COUNT
      WRITE(4,*) (DTFILE(J,L),L=1,13)
  730 CONTINUE
  733 J=J
      WRITE(6,735) J
  735 FORMAT(/2X,I3,' RECORDS WRITTEN TO FILE')
  620 STOP
      END
C *****************************************************
C   SUBROUTINE FOR ENTRY OF TIME DATA
C *****************************************************
      SUBROUTINE TIME(I4,I)
      COMMON DTWORK(50,10),DTFILE(50,13)
      IF (I4.GT.6) GOTO 9
      GOTO (100,200,300,400,500,600),I4
    9 WRITE(6,10) I4
   10 FORMAT(//3X,'*** RESPONSE (',I1,') INVALID. RE-ENTER.')
      RETURN
  100 WRITE(6,105) DTFILE(I,1),DTFILE(I,2)
  105 FORMAT(//3X,'TIME DURATION FOR ARC ',F3.0,1X,F3.0,
     1' : DIST = CONST',/3X,44('-'),//1X,
     2'ENTER CONSTANT TIME DURATION')
      READ(5,*) DTFILE(I,10)
      RETURN
  200 WRITE(6,205) DTFILE(I,1),DTFILE(I,2)
  205 FORMAT(//3X,'TIME DURATION FOR ARC ',F3.0,1X,F3.0,
     1' : DIST = RECT',/3X,43('-'),//1X,
     2'ENTER LOWER BOUND,UPPER BOUND')
      READ(5,*) DTFILE(I,10),DTFILE(I,11)
      RETURN
  300 WRITE(6,305) DTFILE(I,1),DTFILE(I,2)
  305 FORMAT(//3X,'TIME DURATION FOR ARC ',F3.0,1X,F3.0,
     1' : DIST = EXP',/3X,42('-'),//1X,
     2'ENTER MEAN TIME DURATION')
      READ(5,*) DTFILE(I,10)
      RETURN
  400 WRITE(6,405) DTFILE(I,1),DTFILE(I,2)
  405 FORMAT(//3X,'TIME DURATION FOR ARC ',F3.0,1X,F3.0,
     1' : DIST = GAMMA',/3X,43('-'),//1X,
     2'ENTER: 999,MEAN,VARIANCE'/4X,
     3'OR  LOWER BOUND,UPPER BOUND,MODE')
      READ(5,*) (DTFILE(I,L),L=10,12)
      RETURN
  500 WRITE(6,505) DTFILE(I,1),DTFILE(I,2)
  505 FORMAT(//3X,'TIME DURATION FOR ARC ',F3.0,1X,F3.0,
     1' : DIST = NORM',/3X,43('-'),//1X,
     2'ENTER: 999,MEAN,VARIANCE'/4X,
     3'OR  LOWER BOUND,UPPER BOUND,MODE')
      READ(5,*) (DTFILE(I,L),L=10,12)
      RETURN
  600 WRITE(6,605) DTFILE(I,1),DTFILE(I,2)
  605 FORMAT(//3X,'TIME DURATION FOR ARC ',F3.0,1X,F3.0,
     1' : DIST = ARB',/3X,42('-'),//1X,
     2'ENTER: 999,MEAN,VARIANCE'/4X,
```

```
     3'OR  LOWER BOUND,UPPER BOUND,MODE')
      READ(5,*) (DTFILE(I,L),L=10,12)
      RETURN
      END
*****************************************************************
  SUBROUTINE TO DISPLAY DATA
*****************************************************************
      SUBROUTINE EDTDIS(J,ICLASS)
      COMMON DTWORK(50,10),DTFILE(50,13)
      WRITE(6,100)
  100 FORMAT(//3X,'SELECT DISPLAY OPTION',/3X,21('-'),//1X,
     1'1. - FROM NODE',/1X,'2. - TO NODE',/1X,'3. - ALL ARCS',//1X,
     2'ENTER: OPTION,NODE NUMBER (ENTER 3,0 TO DISPLAY ALL ARCS)')
      READ(5,*) J,I4
      IF (J.GT.3) GOTO 119
      GOTO (200,300,400),J
  119 WRITE(6,120) J
  120 FORMAT(//3X,'*** RESPONSE (',I1,') INVALID. RE-ENTER.')
      RETURN
  200 WRITE(6,205) I4
  205 FORMAT(//2X,'DISPLAY OF ARCS FROM ',I3)
      WRITE(6,210)
  210 FORMAT(/2X,25('-'),//6X,
     1'--TERM. CASH FLOW---  --CONT. CASH FLOW---  ---TIME DURATION',
     2' ------',/7X,'999.0  MEAN   VAR.    999.0  MEAN   VAR.    DT',
     3'  999.0 MEAN   VAR.')
      WRITE(6,211)
  211 FORMAT(2X,'TO   LOWER  UPPER  MODE     ',
     1'LOWER  UPPER  MODE        LOWER  UPPER  MODE     PROB')
      WRITE(6,212)
  212 FORMAT(2X,'-----',6('-'),1X,6('-'),2X,6('-'),1X,
     16('-'),1X,6('-'),2X,
     22('-'),1X,6('-'),1X,6('-'),1X,6('-'),2X,4('-'),/)
      DO 250 K=1,50
      KK=INT(DTFILE(K,1))
      IF (KK.NE.I4) GOTO 250
      WRITE(6,240) DTFILE(K,2),(DTFILE(K,L),L=3,13)
  240 FORMAT(F4.0,2X,3(F6.1,1X),3(1X,F6.1),2X,F2.0,3(1X,F6.1),
     13X,F4.2)
  250 CONTINUE
      PAUSE 'PRESS ENTER TO CONTINUE'
      RETURN
  300 WRITE(6,310) I4
  310 FORMAT(/2X,'DISPLAY OF ARC TO ',I3)
      WRITE(6,210)
      WRITE(6,311)
  311 FORMAT(2X,'FROM LOWER  UPPER  MODE     ',
     1'LOWER  UPPER  MODE        LOWER  UPPER  MODE      PROB')
      WRITE(6,212)
      DO 350 K=1,50
      KK = INT(DTFILE(KK,2))
      IF (KK.NE.I4) GOTO 350
      WRITE(6,240) DTFILE(K,1),(DTFILE(K,L),L=3,13)
  350 CONTINUE
      PAUSE 'PRESS ENTER TO CONTINUE'
  400 I5=0
      WRITE(6,210)
      WRITE(6,211)
      WRITE(6,212)
      DO 450 K=1,50
```

```
      IF (I5.LE.10) GOTO 405
      I5=0
      PAUSE 'PRESS ENTER TO CONTINUE'
      WRITE(6,210)
      WRITE(6,211)
      WRITE(6,212)
  405 KK = INT(DTFILE(K,1))
      IF (KK.EQ.0) GOTO 455
      WRITE(6,240) (DTFILE(K,L),L=2,13)
      I5 = I5+1
  450 CONTINUE
  455 RETURN
      END
```

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